Discrete-log attacks
and factorization
Part II

Tanja Lange
Technische Universiteit Eindhoven

14 June 2019

with some slides by
Daniel J. Bernstein

Q sieve

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
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19								
20	2	2					5	

etc.

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Q sieve

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

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2	2
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4	2 2
5	5
6	2 3
7	7
8	2 2 2
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10	2 5
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12	2 2 3
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14	2 7
15	3 5
16	2 2 2 2
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etc.

Q sieve

Sieving
using pri

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Q sieve

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

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9	3 3
10	2 5
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12	2 2 3
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14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

etc.

Q sieve

Sieving i and 611
using primes $2, 3, 5$

1	
2	2
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12	2 2 3
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14	2 7
15	3 5
16	2 2 2 2
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18	2 3 3
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20	2 2 5

612	2
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etc.

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Q sieve

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

i								
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14	2							7
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16	2	2	2	2				
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18	2				3	3		
19								
20	2	2					5	

etc.

Q sieve

Sieving i and $611 + i$ for sm
using primes $2, 3, 5, 7$:

i								
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8	2	2	2					
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10	2						5	
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12	2	2			3			
13								
14	2							7
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16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

$611+i$								
612	2	2			3	3		
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622	2							
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626	2							
627					3			
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etc.

Q sieve

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

1								
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8	2	2	2					
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10	2						5	
11								
12	2	2			3			
13								
14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

etc.

Q sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
5							5	
6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
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631												

etc.

small integers $i > 0$
imes $2, 3, 5, 7$:

	5
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	5
	7
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$\mathbf{Q}$ sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

i	2	3	5	7
1				
2	2			
3		3		
4	2 2			
5			5	
6	2	3		
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8	2 2 2			
9		3 3		
10	2		5	
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12	2 2	3		
13				
14	2			7
15		3	5	
16	2 2 2 2			
17				
18	2	3 3		
19				
20	2 2		5	

$611+i$	2	3	5	7
612	2 2	3 3		
613				
614	2			
615		3	5	
616	2 2 2			7
617				
618	2	3		
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620	2 2		5	
621		3 3 3		
622	2			
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etc.

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the “cor
for some

$14 \cdot 625$

$64 \cdot 675$

$75 \cdot 686$

$14 \cdot 64 \cdot$
$= 2^8 3^4 5$

$\gcd\{611$
$= 47.$

$611 = 4$

gers $i > 0$

5, 7:

Q sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

i	2	3	5	7
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2	2			
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14	2			7
15		3	5	
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18	2	3 3		
19				
20	2 2		5	

$611+i$	2	3	5	7
612	2 2	3 3		
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618	2	3		
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620	2 2		5	
621		3 3 3		
622	2			
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624	2 2 2 2	3		
625			5 5 5 5	
626	2			
627		3		
628	2 2			
629				
630	2	3 3	5	7
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etc.

Have complete fac

the “congruences”

for some i’s.

$14 \cdot 625 = 2^1 3^0 5^4$

$64 \cdot 675 = 2^6 3^3 5^2$

$75 \cdot 686 = 2^1 3^1 5^2$

$14 \cdot 64 \cdot 75 \cdot 625 \cdot$ 6

$= 2^8 3^4 5^8 7^4 = (2^4$

$\gcd\{611, 14 \cdot 64 \cdot$

$= 47.$

$611 = 47 \cdot 13.$

Q sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1				
2	2			
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14	2			7
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614	2			
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624	2 2 2 2	3		
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626	2			
627		3		
628	2 2			
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630	2	3 3	5	7
631				

etc.

Have complete factorization
the “congruences” $i(611 +$
for some i’s.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$
$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2$
$= 47$.

$611 = 47 \cdot 13$.

Q sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1								
2	2							
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etc.

Have complete factorization of the "congruences" $i(611 + i)$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
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$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
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$611 = 47 \cdot 13$.

i and $611 + i$ for small i

imes $2, 3, 5, 7$:

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616	2	2	2									7
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620	2	2						5				
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$611 = 47 \cdot 13$.

Why did

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Have complete factorization of the "congruences" $i(611+i)$ for some i's.

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$611 = 47 \cdot 13$.

Why did this find

Was it just blind l

$\gcd\{611, \text{random}\}$

No.

By construction 61

where $s = 14 \cdot 64$

and $t = 2^4 3^2 5^4 7^2$.

So each prime > 7

divides either $s -$

Not terribly surpris

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Have complete factorization of the "congruences" $i(611+i)$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
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$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

Why did this find a factor o
Was it just blind luck:
$\gcd\{611, \text{random}\} = 47$?

No.

By construction 611 divides
where $s = 14 \cdot 64 \cdot 75$
and $t = 2^4 3^2 5^4 7^2$.
So each prime > 7 dividing
divides either $s - t$ or $s + t$.

Not terribly surprising
(but not guaranteed in adva
that one prime divided $s - t$
and the other divided $s + t$.

Have complete factorization of the "congruences" $i(611 + i)$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
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$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

Why did this find a factor of 611? Was it just blind luck: $\gcd\{611, \text{random}\} = 47$?

No.

By construction 611 divides $s^2 - t^2$ where $s = 14 \cdot 64 \cdot 75$ and $t = 2^4 3^2 5^4 7^2$.
So each prime > 7 dividing 611 divides either $s - t$ or $s + t$.

Not terribly surprising (but not guaranteed in advance!) that one prime divided $s - t$ and the other divided $s + t$.

mplete factorization of
ngruences" $i(611+i)$
e i's.

$= 2^1 3^0 5^4 7^1$.
$= 2^6 3^3 5^2 7^0$.
$= 2^1 3^1 5^2 7^3$.

$75 \cdot 625 \cdot 675 \cdot 686$
${}^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$., 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$

$7 \cdot 13$.

Why did this find a factor of 611?
Was it just blind luck:
$\gcd\{611, \text{random}\} = 47$?

No.

By construction 611 divides $s^2 - t^2$
where $s = 14 \cdot 64 \cdot 75$
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So each prime > 7 dividing 611
divides either $s - t$ or $s + t$.

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(but not guaranteed in advance!)
that one prime divided $s - t$
and the other divided $s + t$.

Why did
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Yes. The
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Why did this find a factor of 611?
Was it just blind luck:
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Not terribly surprising
(but not guaranteed in advance!)
that one prime divided $s - t$
and the other divided $s + t$.

Why did the first t
completely factore
have square produ
Was it just blind l

Yes. The exponen
$(1, 0, 4, 1), (6, 3, 2,$
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But we didn't need
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$5^4 7^2\}$

Why did this find a factor of 611?
Was it just blind luck:
$\gcd\{611, \text{random}\} = 47$?

No.

By construction 611 divides $s^2 - t^2$
where $s = 14 \cdot 64 \cdot 75$
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So each prime > 7 dividing 611
divides either $s - t$ or $s + t$.

Not terribly surprising
(but not guaranteed in advance!)
that one prime divided $s - t$
and the other divided $s + t$.

Why did the first three
completely factored congrue
have square product?
Was it just blind luck?

Yes. The exponent vectors
$(1, 0, 4, 1), (6, 3, 2, 0), (1, 1, 2$
happened to have sum 0 mo

But we didn't need this luck
Given long sequence of vect
easily find nonempty subseq
with sum 0 mod 2.

Why did this find a factor of 611?
Was it just blind luck:
$\gcd\{611, \text{random}\} = 47$?

No.

By construction 611 divides $s^2 - t^2$
where $s = 14 \cdot 64 \cdot 75$
and $t = 2^4 3^2 5^4 7^2$.
So each prime > 7 dividing 611
divides either $s - t$ or $s + t$.

Not terribly surprising
(but not guaranteed in advance!)
that one prime divided $s - t$
and the other divided $s + t$.

Why did the first three
completely factored congruences
have square product?
Was it just blind luck?

Yes. The exponent vectors
$(1, 0, 4, 1), (6, 3, 2, 0), (1, 1, 2, 3)$
happened to have sum 0 mod 2.

But we didn't need this luck!
Given long sequence of vectors,
easily find nonempty subsequence
with sum 0 mod 2.

l this find a factor of 611?
ust blind luck:
, random$\} = 47$?

truction 611 divides $s^2 - t^2$
$= 14 \cdot 64 \cdot 75$
$2^4 3^2 5^4 7^2$.
prime > 7 dividing 611
either $s - t$ or $s + t$.

ibly surprising
guaranteed in advance!)
prime divided $s - t$
other divided $s + t$.

Why did the first three
completely factored congruences
have square product?
Was it just blind luck?

Yes. The exponent vectors
$(1, 0, 4, 1), (6, 3, 2, 0), (1, 1, 2, 3)$
happened to have sum 0 mod 2.

But we didn't need this luck!
Given long sequence of vectors,
easily find nonempty subsequence
with sum 0 mod 2.

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e.g., $1(n$
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11 divides $s^2 - t^2$
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Why did the first three
completely factored congruences
have square product?
Was it just blind luck?

Yes. The exponent vectors
$(1, 0, 4, 1), (6, 3, 2, 0), (1, 1, 2, 3)$
happened to have sum 0 mod 2.

But we didn't need this luck!
Given long sequence of vectors,
easily find nonempty subsequence
with sum 0 mod 2.

This is linear algeb
Guaranteed to find
if number of vecto
exceeds length of

e.g. for $n = 671$:
$1(n + 1) = 2^5 3$
$4(n + 4) = 2^2 3$
$15(n + 15) = 2^1 3$
$49(n + 49) = 2^4 3$
$64(n + 64) = 2^6 3$

$\mathbf{F}_2$-kernel of expon
gen by (0 1 0 1 1)
e.g., $1(n+1)15(n$
is a square.

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$s^2 - t^2$

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Why did the first three completely factored congruences have square product?
Was it just blind luck?

Yes. The exponent vectors $(1, 0, 4, 1), (6, 3, 2, 0), (1, 1, 2, 3)$ happened to have sum 0 mod 2.

But we didn't need this luck!
Given long sequence of vectors, easily find nonempty subsequence with sum 0 mod 2.

This is linear algebra over $\mathbf{F}$
Guaranteed to find subseque
if number of vectors
exceeds length of each vecto

e.g. for $n = 671$:

$$1(n + 1) = 2^5 3^1 5^0 7^1;$$
$$4(n + 4) = 2^2 3^3 5^2 7^0;$$
$$15(n + 15) = 2^1 3^1 5^1 7^3;$$
$$49(n + 49) = 2^4 3^2 5^1 7^2;$$
$$64(n + 64) = 2^6 3^1 5^1 7^2.$$

$\mathbf{F}_2$-kernel of exponent matri
gen by (0 1 0 1 1) and (1 0
e.g., $1(n+1)15(n+15)49(n$
is a square.

Why did the first three completely factored congruences have square product? Was it just blind luck?

Yes. The exponent vectors $(1, 0, 4, 1), (6, 3, 2, 0), (1, 1, 2, 3)$ happened to have sum 0 mod 2.

But we didn't need this luck! Given long sequence of vectors, easily find nonempty subsequence with sum 0 mod 2.

This is linear algebra over $\mathbf{F}_2$. Guaranteed to find subsequence if number of vectors exceeds length of each vector.

e.g. for $n = 671$:

$$
\begin{aligned}
1(n + 1) &= 2^5 3^1 5^0 7^1;\\
4(n + 4) &= 2^2 3^3 5^2 7^0;\\
15(n + 15) &= 2^1 3^1 5^1 7^3;\\
49(n + 49) &= 2^4 3^2 5^1 7^2;\\
64(n + 64) &= 2^6 3^1 5^1 7^2.
\end{aligned}
$$

$\mathbf{F}_2$-kernel of exponent matrix is gen by (0 1 0 1 1) and (1 0 1 1 0); e.g., $1(n+1)15(n+15)49(n+49)$ is a square.

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This is linear algebra over $\mathbf{F}_2$.
Guaranteed to find subsequence
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exceeds length of each vector.

e.g. for $n = 671$:

$$
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1(n+\ 1) &= 2^5 3^1 5^0 7^1;\\
4(n+\ 4) &= 2^2 3^3 5^2 7^0;\\
15(n+15) &= 2^1 3^1 5^1 7^3;\\
49(n+49) &= 2^4 3^2 5^1 7^2;\\
64(n+64) &= 2^6 3^1 5^1 7^2.
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e.g. progression $q - (n$ mod
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etc.

Check smoothness of
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Try many large q's.
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e.g. progression $q - (n \bmod q)$, $2q - (n \bmod q)$, $3q - (n \bmod q)$, etc.

Check smoothness of generalized congruence $i(n+i)/q$ for i's in this sublattice.
e.g. check whether $i, (n+i)/q$ are smooth for $i = q - (n \bmod q)$ etc.

Try many large q's.
Rare for i's to overlap.

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Rare for i's to overlap.

e.g. $n =$

Original

i	n
1	3
2	3
3	3

Use 997^2
$i \in 8024$

8024
17964
27904

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Try many large q's.
Rare for i's to overlap.

e.g. $n = 3141592$

Original $\mathbf{Q}$ sieve:

i	$n+i$
1	314159265
2	314159265
3	314159265

Use 997^2-sublattic

$i \in 802458 + 9940$

i	$(n$
802458	316
1796467	316
2790476	316

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$-i)$

$i \approx y$.

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Choose q, square of large prime.
Choose a "q-sublattice" of i's: arithmetic progression of i's where q divides each $i(n+i)$.
e.g. progression $q-(n \bmod q)$, $2q-(n \bmod q)$, $3q-(n \bmod q)$, etc.

Check smoothness of generalized congruence $i(n+i)/q$ for i's in this sublattice.
e.g. check whether $i, (n+i)/q$ are smooth for $i = q-(n \bmod q)$ etc.

Try many large q's.
Rare for i's to overlap.

e.g. $n = 31415926535897932$

Original $\mathbf{Q}$ sieve:

i	$n+i$
1	31415926535897932
2	31415926535897932
3	31415926535897932

Use 997^2-sublattice,
$i \in 802458 + 994009\mathbf{Z}$:

i	$(n+i)/997^2$
802458	316052737309
1796467	316052737310
2790476	316052737311

Choose q, square of large prime.
Choose a "q-sublattice" of i's: arithmetic progression of i's where q divides each $i(n+i)$.
e.g. progression $q - (n \bmod q)$, $2q - (n \bmod q)$, $3q - (n \bmod q)$, etc.

Check smoothness of generalized congruence $i(n+i)/q$ for i's in this sublattice.
e.g. check whether $i, (n+i)/q$ are smooth for $i = q - (n \bmod q)$ etc.

Try many large q's.
Rare for i's to overlap.

e.g. $n = 314159265358979323$:

Original $\mathbf{Q}$ sieve:

i	$n+i$
1	314159265358979324
2	314159265358979325
3	314159265358979326

Use 997^2-sublattice,
$i \in 802458 + 994009\mathbf{Z}$:

i	$(n+i)/997^2$
802458	316052737309
1796467	316052737310
2790476	316052737311

q, square of large prime.
a "q-sublattice" of i's:
tic progression of i's
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$\bmod q)$, $3q-(n \bmod q)$,

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$458 + 994009\mathbf{Z}$:

i	$(n+i)/997^2$
458	316052737309
467	316052737310
476	316052737311

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"Quadratic sieve" (QS) uses
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Generalizing beyond $\mathbf{Q}$

The $\mathbf{Q}$ sieve is a special cas the number-field sieve.

Recall how the $\mathbf{Q}$ sieve factors 611:

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Form a square

as product of $(i +$

for several pairs $(i$

$(-11 + 3 \cdot 25)(-1$

$\cdot\, (3 + 25)(3$

$= (112 - 16\sqrt{14})^2$

Compute

$s = (-11 + 3 \cdot 25)$

$t = 112 - 16 \cdot 25,$

$\gcd\{611, s - t\} =$

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Form a square
as product of $(i + 25j)(i +$
for several pairs (i, j):
$(-11 + 3 \cdot 25)(-11 + 3\sqrt{14}$
$\cdot (3 + 25)(3 + \sqrt{14})$
$= (112 - 16\sqrt{14})^2$.

Compute
$s = (-11 + 3 \cdot 25) \cdot (3 + 25$
$t = 112 - 16 \cdot 25$,
$\gcd\{611, s - t\} = 13$.

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Form a square as product of $(i+25j)(i+\sqrt{14}j)$ for several pairs (i,j):
$(-11+3\cdot 25)(-11+3\sqrt{14})$
$\quad\cdot(3+25)(3+\sqrt{14})$
$= (112-16\sqrt{14})^2$.

Compute
$s = (-11+3\cdot 25)\cdot(3+25)$,
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Answer:
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Compute
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Why does this wor

Answer: Have ring
$\mathbf{Z}[\sqrt{14}] \to \mathbf{Z}/611,$
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Apply ring morphi

$$(-11 + 3 \cdot 25)(-1 \\ \cdot (3 + 25)(3 \\ = (112 - 16 \cdot 25)^2$$

i.e. $s^2 = t^2$ in $\mathbf{Z}/6$

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$(-11 + 3 \cdot 25)(-11 + 3\sqrt{14})$
$\quad \cdot (3 + 25)(3 + \sqrt{14})$
$= (112 - 16\sqrt{14})^2.$

Compute
$s = (-11 + 3 \cdot 25) \cdot (3 + 25)$,
$t = 112 - 16 \cdot 25$,
$\gcd\{611, s - t\} = 13.$

Why does this work?

Answer: Have ring morphisr
$\mathbf{Z}[\sqrt{14}] \to \mathbf{Z}/611$, $\sqrt{14} \mapsto 2$
since $25^2 = 14$ in $\mathbf{Z}/611$.

Apply ring morphism to squa
$(-11 + 3 \cdot 25)(-11 + 3 \cdot 25$
$\quad \cdot (3 + 25)(3 + 25)$
$= (112 - 16 \cdot 25)^2$ in $\mathbf{Z}/611$

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Unsurprising to find factor.

Generali

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$m \in \mathbf{Z}$,

Write d

$f = f_d x$

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Pick $r \in$

Then f_d

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Unsurprising to find factor.

Generalize from $(x$

to (f,m) with irre

$m\in\mathbf{Z}$, $f(m)\in n$

Write $d=\deg f$,

$f\;=f_dx^d+\cdots+$

Can take $f_d=1$ f

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Pick $r\in\mathbf{C}$, root o

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monic $g=f_d^{d-1}f($

$\mathbf{Q}(r)\leftarrow\mathcal{O}\leftarrow\mathbf{Z}[f_dr]$

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$= (112 - 16 \cdot 25)^2$ in $\mathbf{Z}/611$.

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Generalize from $(x^2 - 14, 25$
to (f, m) with irred $f \in \mathbf{Z}[x$
$m \in \mathbf{Z}$, $f(m) \in n\mathbf{Z}$.

Write $d = \deg f$,
$f = f_d x^d + \cdots + f_1 x^1 + f_0$

Can take $f_d = 1$ for simplici
but larger f_d allows
better parameter selection.

Pick $r \in \mathbf{C}$, root of f.
Then $f_d r$ is a root of
monic $g = f_d^{d-1} f(x/f_d) \in \mathbf{Z}$

$\mathbf{Q}(r) \leftarrow \mathcal{O} \leftarrow \mathbf{Z}[f_d r] \xrightarrow{f_d r \mapsto f_d m}$

Why does this work?

Answer: Have ring morphism $\mathbf{Z}[\sqrt{14}] \to \mathbf{Z}/611$, $\sqrt{14} \mapsto 25$, since $25^2 = 14$ in $\mathbf{Z}/611$.

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Have ring morphism

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ng morphism to square:

$3 \cdot 25)(-11 + 3 \cdot 25)$

$3 + 25)(3 + 25)$

$- 16 \cdot 25)^2$ in $\mathbf{Z}/611$.

$= t^2$ in $\mathbf{Z}/611$.

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Generalize from $(x^2 - 14, 25)$ to (f, m) with irred $f \in \mathbf{Z}[x]$, $m \in \mathbf{Z}$, $f(m) \in n\mathbf{Z}$.

Write $d = \deg f$, $f = f_d x^d + \cdots + f_1 x^1 + f_0 x^0$.

Can take $f_d = 1$ for simplicity, but larger f_d allows better parameter selection.

Pick $r \in \mathbf{C}$, root of f. Then $f_d r$ is a root of monic $g = f_d^{d-1} f(x/f_d) \in \mathbf{Z}[x]$.

$$\mathbf{Q}(r) \leftarrow \mathcal{O} \leftarrow \mathbf{Z}[f_d r] \xrightarrow{f_d r \mapsto f_d m} \mathbf{Z}/n$$

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Then apply the ring morphism $\varphi : \mathbf{Z}[f_d r] \to \mathbf{Z}/n$ taking $f_d r$ to $f_d m$. Compute $\gcd\{n, \varphi(r) - g'(f_d m)\prod(i - j\,m)f_d\}$. In $\mathbf{Z}/n$ have $\varphi(r)^2 = g'(f_d m)^2 \prod(i - j\,m)^2 f_d^2$.

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How to find squar
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Look for y-smooth
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How to find square product
of congruences $(i - j\,m)(i -$

Start with congruences for,
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Look for y-smooth congruen
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$f_d i^d + \cdots + f_0 j^d = j^d f(i/j$
Norm covers all d roots r.
Here "y-smooth" means
"has no prime divisor $> y$."

Find enough smooth congru
Perform linear algebra on
exponent vectors mod 2.

$\prod(i - jm)(i - jr)f_d^2$
is a square in $\mathcal{O}$,
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Find enough smooth congruences.
Perform linear algebra on exponent vectors mod 2.

Polynomial selecti

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General strategy:
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Polynomial selection

Many f's possible for n.
How to find f that
minimizes NFS time?

General strategy:
Enumerate many f's.
For each f, estimate time us
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How to find square product of congruences $(i - jm)(i - jr)$?

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Let's restrict atten

$(x - m)(f_5 x^5 + f$

Take m near $n^{1/6}$

Expand n in base

$n = f_5 m^5 + f_4 m^4$

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Have $f_5 \approx n^{1/6}$.

Typically all the f_i

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For each f, estimate time using information about f arithmetic, distribution of $j^{\deg f} f(i/j)$, distribution of smooth numbers.

Let's restrict attention to $f($
$(x - m)(f_5 x^5 + f_4 x^4 + \cdots$

Take m near $n^{1/6}$.
Expand n in base m:
$n = f_5 m^5 + f_4 m^4 + \cdots + f$
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Have $f_5 \approx n^{1/6}$.
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Have $f_5 \approx n^{1/6}$.
Typically all the f_i's
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(1993 Buhler Lenstra Pomerance)

To reduce f value

Enumerate many p
for m near $B^{0.25} n$

Have $f_5 \approx B^{-1.25}$
f_4, f_3, f_2, f_1, f_0 co
as large as $B^{0.25} n$
Hope that they ar
on scale of $B^{-1.25}$

Conjecturally this
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Let's restrict attention to $f(x) = (x - m)(f_5x^5 + f_4x^4 + \cdots + f_0)$.

Take m near $n^{1/6}$.
Expand n in base m:
$n = f_5m^5 + f_4m^4 + \cdots + f_0$.
Can use negative coefficients.

Have $f_5 \approx n^{1/6}$.
Typically all the f_i's
are on scale of $n^{1/6}$.

(1993 Buhler Lenstra Pomerance)

To reduce f values by factor

Enumerate many possibilitie
for m near $B^{0.25}n^{1/6}$.

Have $f_5 \approx B^{-1.25}n^{1/6}$.
f_4, f_3, f_2, f_1, f_0 could be
as large as $B^{0.25}n^{1/6}$.
Hope that they are smaller,
on scale of $B^{-1.25}n^{1/6}$.

Conjecturally this happens
within roughly $B^{7.5}$ trials.
Then $(i - jm)(f_5i^5 + \cdots +$
is on scale of $B^{-1}R^6n^{2/6}$
for i, j on scale of R.
Several more ways; depends

Let's restrict attention to $f(x) = (x - m)(f_5 x^5 + f_4 x^4 + \cdots + f_0)$.

Take m near $n^{1/6}$.
Expand n in base m:
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Can use negative coefficients.

Have $f_5 \approx n^{1/6}$.
Typically all the f_i's
are on scale of $n^{1/6}$.

(1993 Buhler Lenstra Pomerance)

To reduce f values by factor B:

Enumerate many possibilities
for m near $B^{0.25} n^{1/6}$.

Have $f_5 \approx B^{-1.25} n^{1/6}$.
f_4, f_3, f_2, f_1, f_0 could be
as large as $B^{0.25} n^{1/6}$.
Hope that they are smaller,
on scale of $B^{-1.25} n^{1/6}$.

Conjecturally this happens
within roughly $B^{7.5}$ trials.
Then $(i - jm)(f_5 i^5 + \cdots + f_0 j^5)$
is on scale of $B^{-1} R^6 n^{2/6}$
for i, j on scale of R.
Several more ways; depends on n.

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$(f_5 x^5 + f_4 x^4 + \cdots + f_0)$.

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Test smoothness of $i - jm$
for all coprime pairs (i, j)
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For each (i, j)
with smooth $i - jm$,
test smoothness of $i - jr$
and $i - j\beta$ and so on,
using primes $\le L^{0.82\ldots+o(1)}$.

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Each $|j^d f(i/j)| \le m^{2.86\ldots+}$
Conjecturally $L^{0.95\ldots+o(1)}$
smooth congruences.

$L^{0.95\ldots+o(1)}$ components
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Many n's can share one m;
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Many n's can share one m; $L^{1.90\ldots+o(1)}$ bit operations to find squares for *all* n's.

Oops, linear algebra hurts; fix by reducing y.
But still end up factoring batch in much less time than factoring each n separately.

Asymptotic batch-NFS parameters:

$d/(\log n)^{1/3}(\log\log n)^{-1/3} \in 1.10\ldots + o(1)$.

Primes $\le L^{0.82\ldots+o(1)}$.

$1 \le i, j \le L^{1.00\ldots+o(1)}$.

Computation independent of
finds $L^{1.64\ldots+o(1)}$ smooth values $i - jm$.

$L^{1.64\ldots+o(1)}$ operations for each target n.

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Batch NFS for RS

Expand n in base
$n = n_7 m^7 + n_6 m$
with $0 \le n_0, n_1, .$

Assume irreducibil
$n_7 x^7 + n_6 x^6 + \cdot\cdot$

Choose height H
consider pairs $(a,$
that $-H \le a \le H$
and $\gcd\{a, b\} = 1$

Choose smoothnes
$y = 2^{66} + 2^{55}$.

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Asymptotic batch-NFS parameters:

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Computation independent of n
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smooth values $i - jm$.

$L^{1.64\ldots+o(1)}$ operations
for each target n.

Batch NFS for RSA-3072

Expand n in base $m = 2^{384}$
$n = n_7 m^7 + n_6 m^6 + \cdots +$
with $0 \le n_0, n_1, \ldots, n_7 < m$

Assume irreducibility of
$n_7 x^7 + n_6 x^6 + \cdots + n_0$.

Choose height $H = 2^{62} + 2^{61}$
consider pairs $(a, b) \in \mathbf{Z} \times \mathbf{Z}$
that $-H \le a \le H$, $0 < b \le$
and $\gcd\{a, b\} = 1$.

Choose smoothness bound
$y = 2^{66} + 2^{55}$.

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with $0 \le n_0, n_1, \ldots, n_7 < m$.

Assume irreducibility of
$n_7 x^7 + n_6 x^6 + \cdots + n_0$.

Choose height $H = 2^{62}+2^{61}+2^{57}$: consider pairs $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ such that $-H \le a \le H$, $0 < b \le H$, and $\gcd\{a, b\} = 1$.

Choose smoothness bound $y = 2^{66} + 2^{55}$.

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Find all pairs $(a, b$
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Batch NFS for RSA-3072

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There are about
$12H^2/\pi^2 \approx 2^{125.51}$
pairs (a, b).

Find all pairs (a, b) with
y-smooth $(a - bm)c$ where
$c = n_7 a^7 + n_6 a^6 b + \cdots + n$

Combine these congruences
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Batch NFS for RSA-3072

Expand n in base $m = 2^{384}$:
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Have $u \approx 6.707$
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Heuristic approximation:
$a - bm$ has same y-smooth
chance as a uniform random
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and this chance is u^{-u}
where $u = (\log(Hm))/\log y$

Have $u \approx 6.707$
and $u^{-u} \approx 2^{-18.42}$,
so there are about
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Have $v \approx 12.395$ and $v^{-v} \approx 2^{-45.01}$, so there are about $2^{62.08}$ pairs (a, b) such that $a - bm$ and c are both smoo

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Biggest step in computation:
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Biggest step in computation:
Check $2^{125.51}$ pairs (a, b) to find the $2^{107.09}$ pairs where $a - bm$ is smooth.

This step is independent of N, reused by many integers N.

Biggest step depending on N:
Check $2^{107.09}$ pairs (a, b) to see whether c is smooth.

This is much less computation! . . . or is it?

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Fix: factor each number
separately:
start with trial division,
then Pollard rho,
then Pollard $p - 1$,
then ECM.

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http://facthacks.cr.yp.to/

The rho

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$(\rho_1 - \rho_2$
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The $2^{107.09}$ pairs (a, b)
are not consecutive,
so no easy way to sieve
for prime divisors of c.

Fix: factor each number
separately:
start with trial division,
then Pollard rho,
then Pollard $p-1$,
then ECM.

Most of them covered in
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The rho method

Define $\rho_0 = 0$, ρ_{k-}

Every prime $\leq 2^{20}$
$(\rho_1 - \rho_2)(\rho_2 - \rho_4$
$\cdots(\rho_{3575} - \rho_{7150})$
Also many larger p

Can compute gcd{
$\approx 2^{14}$ multiplicati
very little memory

Compare to $\approx 2^{16}$
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The rho method

Define $\rho_0 = 0$, $\rho_{k+1} = \rho_k^2 +$

Every prime $\leq 2^{20}$ divides S
$(\rho_1 - \rho_2)(\rho_2 - \rho_4)(\rho_3 - \rho_6)$
$\cdots(\rho_{3575} - \rho_{7150})$.
Also many larger primes.

Can compute $\gcd\{c, S\}$ usin
$\approx 2^{14}$ multiplications mod c
very little memory.

Compare to $\approx 2^{16}$ divisions
for trial division up to 2^{20}.

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Fix: factor each number separately:
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then Pollard $p-1$,
then ECM.

Most of them covered in http://facthacks.cr.yp.to/

The rho method

Define $\rho_0 = 0$, $\rho_{k+1} = \rho_k^2 + 11$.

Every prime $\le 2^{20}$ divides $S = (\rho_1 - \rho_2)(\rho_2 - \rho_4)(\rho_3 - \rho_6) \cdots (\rho_{3575} - \rho_{7150})$.
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Can compute $\gcd\{c, S\}$ using $\approx 2^{14}$ multiplications mod c, very little memory.

Compare to $\approx 2^{16}$ divisions for trial division up to 2^{20}.

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Compare to $\approx 2^{16}$ divisions
for trial division up to 2^{20}.

More generally: C
Compute $\gcd\{c, S$
$(\rho_1 - \rho_2)(\rho_2 - \rho_4$

How big does z ha
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Plausible conjectu
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Reason: Consider
$\rho_1 \bmod p, \rho_2 \bmod p$
If $\rho_i \bmod p = \rho_j$ m
then $\rho_k \bmod p = \rho$
for $k \in (j - i)\mathbf{Z} \cap$

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Also many larger primes.

Can compute $\gcd\{c, S\}$ using $\approx 2^{14}$ multiplications mod c, very little memory.

Compare to $\approx 2^{16}$ divisions for trial division up to 2^{20}.

More generally: Choose z.
Compute $\gcd\{c, S\}$ where S
$(\rho_1 - \rho_2)(\rho_2 - \rho_4)\cdots(\rho_z -$

How big does z have to be
for all primes $\le y$ to divide

Plausible conjecture: $y^{1/2+o}$
so $y^{1/2+o(1)}$ mults mod c.

Reason: Consider first collis
$\rho_1 \bmod p, \rho_2 \bmod p, \ldots$.
If $\rho_i \bmod p = \rho_j \bmod p$
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How big does z have to be
for all primes $\le y$ to divide S?

Plausible conjecture: $y^{1/2+o(1)}$;
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Reason: Consider first collision in
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The $p -$

$S_1 = 2^{2}$
divisors
3, 5, 7,
37, 41, 4
89, 97, 1
137, 151

These di
70 of th
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etc.

$_{+1} = \rho_k^2 + 11.$

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o to 2^{20}.

More generally: Choose z.

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for $k \in (j - i)\mathbf{Z} \cap [i, \infty] \cap [j, \infty]$.

<u>The $p - 1$ method</u>

$S_1 = 2^{232792560}$ –
divisors
3, 5, 7, 11, 13, 17
37, 41, 43, 53, 61,
89, 97, 103, 109, 1
137, 151, 157, 181

These divisors incl
70 of the 168 prim
156 of the 1229 pr
296 of the 9592 pr
470 of the 78498 p
etc.

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More generally: Choose z.
Compute $\gcd\{c, S\}$ where $S = (\rho_1 - \rho_2)(\rho_2 - \rho_4)\cdots(\rho_z - \rho_{2z})$.

How big does z have to be for all primes $\le y$ to divide S?

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The $p - 1$ method

$S_1 = 2^{232792560} - 1$ has pr
divisors
3, 5, 7, 11, 13, 17, 19, 23, 2
37, 41, 43, 53, 61, 67, 71, 7
89, 97, 103, 109, 113, 127,
137, 151, 157, 181, 191, 19

These divisors include
70 of the 168 primes $\le 10^3$;
156 of the 1229 primes $\le$ 10
296 of the 9592 primes $\le$ 10
470 of the 78498 primes $\le$ 1
etc.

More generally: Choose z.
Compute $\gcd\{c, S\}$ where $S = (\rho_1 - \rho_2)(\rho_2 - \rho_4)\cdots(\rho_z - \rho_{2z})$.

How big does z have to be for all primes $\leq y$ to divide S?

Plausible conjecture: $y^{1/2+o(1)}$; so $y^{1/2+o(1)}$ mults mod c.

Reason: Consider first collision in $\rho_1 \bmod p, \rho_2 \bmod p, \ldots$.
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then $\rho_k \bmod p = \rho_{2k} \bmod p$
for $k \in (j - i)\mathbf{Z} \cap [i, \infty] \cap [j, \infty]$.

<u>The $p-1$ method</u>

$S_1 = 2^{232792560} - 1$ has prime divisors
3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 53, 61, 67, 71, 73, 79, 89, 97, 103, 109, 113, 127, 131, 137, 151, 157, 181, 191, 199 etc.

These divisors include
70 of the 168 primes $\leq 10^3$;
156 of the 1229 primes $\leq 10^4$;
296 of the 9592 primes $\leq 10^5$;
470 of the 78498 primes $\leq 10^6$;
etc.

nerally: Choose z.

e $\gcd\{c, S\}$ where $S =$

$)(\rho_2 - \rho_4) \cdots (\rho_z - \rho_{2z})$.

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rimes $\le y$ to divide S?

e conjecture: $y^{1/2+o(1)}$;

$-o(1)$ mults mod c.

Consider first collision in

$p, \rho_2 \bmod p, \ldots$.

d $p = \rho_j \bmod p$

mod $p = \rho_{2k} \bmod p$

$(j - i)\mathbf{Z} \cap [i, \infty] \cap [j, \infty]$.

<u>The $p - 1$ method</u>

$S_1 = 2^{232792560} - 1$ has prime divisors
3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 53, 61, 67, 71, 73, 79, 89, 97, 103, 109, 113, 127, 131, 137, 151, 157, 181, 191, 199 etc.

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etc.

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Answer:

$= \mathrm{lcm}\{1$

$= 2^4 \cdot 3^2$

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$) \cdots (\rho_z - \rho_{2z})$.

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$[i, \infty] \cap [j, \infty]$.

The $p-1$ method

$S_1 = 2^{232792560} - 1$ has prime divisors
3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 53, 61, 67, 71, 73, 79, 89, 97, 103, 109, 113, 127, 131, 137, 151, 157, 181, 191, 199 etc.

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etc.

An odd prime p
divides $2^{232792560}$
iff order of 2 in th
multiplicative grou
divides $s = 232792$

Many ways for thi
232792560 has 96

Why so many?
Answer: $s = 2327$
$= \mathrm{lcm}\{1, 2, 3, 4, 5,$
$= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 1$

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The $p - 1$ method

$S_1 = 2^{232792560} - 1$ has prime divisors
3, 5, 7, 11, 13, 17, 19, 23, 29, 31,
37, 41, 43, 53, 61, 67, 71, 73, 79,
89, 97, 103, 109, 113, 127, 131,
137, 151, 157, 181, 191, 199 etc.

These divisors include
70 of the 168 primes $\le 10^3$;
156 of the 1229 primes $\le 10^4$;
296 of the 9592 primes $\le 10^5$;
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etc.

An odd prime p divides $2^{232792560} - 1$ iff order of 2 in the multiplicative group $\mathbf{F}_p^*$ divides $s = 232792560$.

Many ways for this to happe
232792560 has 960 divisors.

Why so many?
Answer: $s = 232792560$
$= \mathrm{lcm}\{1, 2, 3, 4, 5, \ldots, 20\}$
$= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot$

The $p-1$ method

$S_1 = 2^{232792560} - 1$ has prime divisors
3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 53, 61, 67, 71, 73, 79, 89, 97, 103, 109, 113, 127, 131, 137, 151, 157, 181, 191, 199 etc.

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An odd prime p divides $2^{232792560} - 1$ iff order of 2 in the multiplicative group $\mathbf{F}_p^*$ divides $s = 232792560$.

Many ways for this to happen: 232792560 has 960 divisors.

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Answer: $s = 232792560$
$= \text{lcm}\{1, 2, 3, 4, 5, \ldots, 20\}$
$= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$

1 method

$232792560 - 1$ has prime

11, 13, 17, 19, 23, 29, 31,
43, 53, 61, 67, 71, 73, 79,
103, 109, 113, 127, 131,
., 157, 181, 191, 199 etc.

ivisors include
e 168 primes $\leq 10^3$;
he 1229 primes $\leq 10^4$;
he 9592 primes $\leq 10^5$;
he 78498 primes $\leq 10^6$;

An odd prime p
divides $2^{232792560} - 1$
iff order of 2 in the
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divides $s = 232792560$.

Many ways for this to happen:
232792560 has 960 divisors.

Why so many?
Answer: $s = 232792560$
$= \text{lcm}\{1, 2, 3, 4, 5, \ldots, 20\}$
$= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

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$2^2 = 2 \cdot$
$2^{12} = 2^6$
$2^{55}; 2^{110}$
$2^{3552}; 2^7$
$2^{56834}; 2^1$
$2^{909345};$
$2^{3637383}$
$2^{14549535}$
$2^{11639628}$

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, 19, 23, 29, 31,
, 67, 71, 73, 79,
113, 127, 131,
1, 191, 199 etc.

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nes $\leq 10^3$;
rimes $\leq 10^4$;
rimes $\leq 10^5$;
primes $\leq 10^6$;

An odd prime p
divides $2^{232792560} - 1$
iff order of 2 in the
multiplicative group $\mathbf{F}_p^*$
divides $s = 232792560$.

Many ways for this to happen:
232792560 has 960 divisors.

Why so many?
Answer: $s = 232792560$
$= \operatorname{lcm}\{1, 2, 3, 4, 5, \ldots, 20\}$
$= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

Can compute 2^{232}
using 41 ring oper
(Side note: 41 is n

Ring operation: 0,

This computation:
$2^2 = 2 \cdot 2$; $2^3 = 2^2$
$2^{12} = 2^6 \cdot 2^6$; $2^{13} =$
2^{55}; 2^{110}; 2^{111}; 2^{222}
2^{3552}; 2^{7104}; 2^{14208}
2^{56834}; 2^{113668}; 2^{2273}
2^{909345}; $2^{1818690}$; 2
$2^{3637383}$; $2^{7274766}$;
$2^{14549535}$; $2^{2909907}$
$2^{116396280}$; $2^{2327925}$

ime

29, 31,
73, 79,
131,
9 etc.

0^4;
0^5;
10^6;

An odd prime p divides $2^{232792560} - 1$ iff order of 2 in the multiplicative group $\mathbf{F}_p^*$ divides $s = 232792560$.

Many ways for this to happen: 232792560 has 960 divisors.

Why so many?
Answer: $s = 232792560$
$= \text{lcm}\{1, 2, 3, 4, 5, \ldots, 20\}$
$= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

Can compute $2^{232792560} - 1$
using 41 ring operations.
(Side note: 41 is not minim

Ring operation: 0, 1, $+$, $-$,

This computation: 1; $2 = 1$
$2^2 = 2 \cdot 2$; $2^3 = 2^2 \cdot 2$; $2^6 =$
$2^{12} = 2^6 \cdot 2^6$; $2^{13} = 2^{12} \cdot 2$; 2^{26};
2^{55}; 2^{110}; 2^{111}; 2^{222}; 2^{444}; 2^{88}
2^{3552}; 2^{7104}; 2^{14208}; 2^{28416}; 2
2^{56834}; 2^{113668}; 2^{227336}; 2^{454672};
2^{909345}; $2^{1818690}$; $2^{1818691}$; 2^{3}
$2^{3637383}$; $2^{7274766}$; $2^{7274767}$; 2
$2^{14549535}$; $2^{29099070}$; $2^{5819814}$
$2^{116396280}$; $2^{232792560}$; 2^{23279}

An odd prime p
divides $2^{232792560} - 1$
iff order of 2 in the
multiplicative group $\mathbf{F}_p^*$
divides $s = 232792560$.

Many ways for this to happen:
232792560 has 960 divisors.

Why so many?
Answer: $s = 232792560$
$= \operatorname{lcm}\{1, 2, 3, 4, 5, \ldots, 20\}$
$= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$

Can compute $2^{232792560} - 1$
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(Side note: 41 is not minimal.)

Ring operation: 0, 1, $+$, $-$, $\cdot$.

This computation: 1; $2 = 1 + 1$;
$2^2 = 2 \cdot 2$; $2^3 = 2^2 \cdot 2$; $2^6 = 2^3 \cdot 2^3$;
$2^{12} = 2^6 \cdot 2^6$; $2^{13} = 2^{12} \cdot 2$; 2^{26}; 2^{27}; 2^{54};
2^{55}; 2^{110}; 2^{111}; 2^{222}; 2^{444}; 2^{888}; 2^{1776};
2^{3552}; 2^{7104}; 2^{14208}; 2^{28416}; 2^{28417};
2^{56834}; 2^{113668}; 2^{227336}; 2^{454672}; 2^{909344};
2^{909345}; $2^{1818690}$; $2^{1818691}$; $2^{3637382}$;
$2^{3637383}$; $2^{7274766}$; $2^{7274767}$; $2^{14549534}$;
$2^{14549535}$; $2^{29099070}$; $2^{58198140}$;
$2^{116396280}$; $2^{232792560}$; $2^{232792560} - 1$.

prime p

$2^{232792560} - 1$

of 2 in the

cative group $\mathbf{F}_p^*$

$s = 232792560$.

ays for this to happen:

60 has 960 divisors.

many?

$s = 232792560$

$, 2, 3, 4, 5, \ldots, 20\}$

$^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

Can compute $2^{232792560} - 1$
using 41 ring operations.
(Side note: 41 is not minimal.)

Ring operation: $0, 1, +, -, \cdot$.

This computation: 1; $2 = 1 + 1$;
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2^{55}; 2^{110}; 2^{111}; 2^{222}; 2^{444}; 2^{888}; 2^{1776};
2^{3552}; 2^{7104}; 2^{14208}; 2^{28416}; 2^{28417};
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2^{909345}; $2^{1818690}$; $2^{1818691}$; $2^{3637382}$;
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$2^{14549535}$; $2^{29099070}$; $2^{58198140}$;
$2^{116396280}$; $2^{232792560}$; $2^{232792560} - 1$.

Given po

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using 41

Notation

e.g. $c =$

2^{27} mod

2^{54} mod

2^{55} mod

2^{110} mod

$2^{2327925}$

$-\ 1$

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up $\mathbf{F}_p^*$

2560.

s to happen:

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$, \ldots, 20\}$

$1 \cdot 13 \cdot 17 \cdot 19.$

Can compute $2^{232792560} - 1$
using 41 ring operations.
(Side note: 41 is not minimal.)

Ring operation: 0, 1, $+$, $-$, $\cdot$.

This computation: 1; $2 = 1 + 1$;
$2^2 = 2 \cdot 2$; $2^3 = 2^2 \cdot 2$; $2^6 = 2^3 \cdot 2^3$;
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2^{55}; 2^{110}; 2^{111}; 2^{222}; 2^{444}; 2^{888}; 2^{1776};
2^{3552}; 2^{7104}; 2^{14208}; 2^{28416}; 2^{28417};
2^{56834}; 2^{113668}; 2^{227336}; 2^{454672}; 2^{909344};
2^{909345}; $2^{1818690}$; $2^{1818691}$; $2^{3637382}$;
$2^{3637383}$; $2^{7274766}$; $2^{7274767}$; $2^{14549534}$;
$2^{14549535}$; $2^{29099070}$; $2^{58198140}$;
$2^{116396280}$; $2^{232792560}$; $2^{232792560} - 1$.

Given positive inte
can compute 2^{2327}
using 41 operation
Notation: $a \bmod b$

e.g. $c = 85972312$
$2^{27} \bmod c = 1342$
$2^{54} \bmod c = 1342$
$= 9356$
$2^{55} \bmod c = 1871$
$2^{110} \bmod c = 1871$
$= 1458$
$2^{232792560} - 1 \bmod$

en:

19.

Can compute $2^{232792560} - 1$
using 41 ring operations.
(Side note: 41 is not minimal.)

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This computation: 1; $2 = 1 + 1$;
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Given positive integer n,
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Given positive integer n,
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Scale up
$s = \text{lcm}$
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Scale up to larger
$s = \mathrm{lcm}\{1, 2, 3, 4,$
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find 2317 of the p

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Scale up to larger exponent

$s = \operatorname{lcm}\{1, 2, 3, 4, 5, \ldots, 100$

using 136 squarings mod c

find 2317 of the primes ≤ 1

Is a squaring mod c

faster than 17 trial divisions

Or

$s = \operatorname{lcm}\{1, 2, 3, 4, 5, \ldots, 100$

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find 180121 of the primes $\leq$

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Extra benefit:

no need to store the primes.

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Scale up to larger exponent $s = \mathrm{lcm}\{1, 2, 3, 4, 5, \ldots, 100\}$: using 136 squarings mod c find 2317 of the primes $\le 10^5$.

Is a squaring mod c faster than 17 trial divisions?

Or $s = \mathrm{lcm}\{1, 2, 3, 4, 5, \ldots, 1000\}$: using 1438 squarings mod c find 180121 of the primes $\le 10^7$.

Is a squaring mod c faster than 125 trial divisions?

Extra benefit: no need to store the primes.

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then $p-1$ divides $\operatorname{lcm}\{1,2,\ldots$
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Same if $p-1$ is replaced by
order of 2 in $\mathbf{F}_p^*$.

So uniform random prime p
divides $2^{\operatorname{lcm}\{1,2,\ldots,K\}}-1$
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Similar time spent on trial division
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Safe primes

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Given an integer c, compute $5^{232792560} X$ mod c and compute gcd with c, hoping to factor c.

Many p's not found by $\mathbf{F}_p^*$ are found by $\text{Clock}(\mathbf{F}_p)$.

If -1 is not a square mod p and $p+1$ divides 23279256 then $5^{232792560} X \bmod p = 0$

Proof: $p \equiv 3 \pmod 4$, so $(4/5 + 3i/5)^p = 4/5 - 3i/5$ so $(p+1)(3/5, 4/5) = (0, 1$ in the group $\text{Clock}(\mathbf{F}_p)$ so $232792560(3/5, 4/5) = ($

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The elliptic-curve

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The elliptic-curve method

Stage 1: Point P on E over compute $R = sP$ for $s = \operatorname{lcm}\{2,3,\ldots,B_1\}$.

Stage 2: Small primes $B_1 < q_1, \ldots, q_k \le B_2$ compute $R_i = q_i R$.

If order of P on $E/\mathbf{F}_{p_i}$ (same curve, reduce mod p_i divides sq_i, then $R_i = (0,1)$ (using Edwards)

Compute $\gcd\{c, \prod y(R_i)\}$.

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The elliptic-curve method

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Plausible conjecture: if B_1 is $\exp\sqrt{\left(\frac{1}{2} + o(1)\right)\log H \log\log H}$ then, for each prime $p \le H$, a uniform random curve mod p has chance $\ge 1/B_1^{1+o(1)}$ to find p.

Find p using, $\le B_1^{1+o(1)}$ curves; $\le B_1^{2+o(1)}$ squarings. Time subexponential in H.

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Bad RSA randomness

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<u>Bad RSA randomness</u>

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2012.02.14 Lenstra–Hughes–Augier–Bos–Kleinjung–Wachter "Ron was wrong, Whit is right" (Crypto 2012): checked $7 \cdot 10^6$ SSL/PGP RSA keys; found $6 \cdot 10^6$ distinct keys; factored 12720 of those,
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checked $>10^7$ SSL/SSH RSA keys; factored 24816 SSL keys, 2422 SSH host keys.

"Almost all of the vulnerable keys were generated by and are used to secure embedded hardware devices such as routers and firewalls, not to secure popular web sites such as your bank or email provider."

These computations find q_2
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Do this with Taiwan citizen
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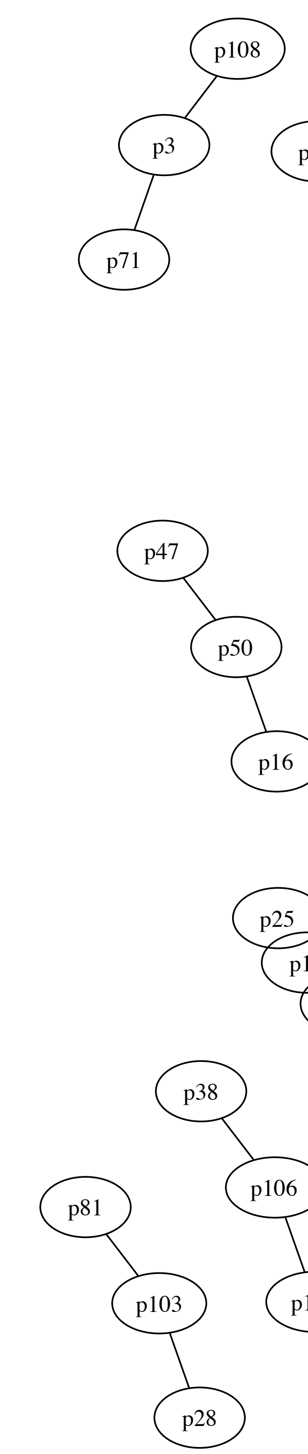

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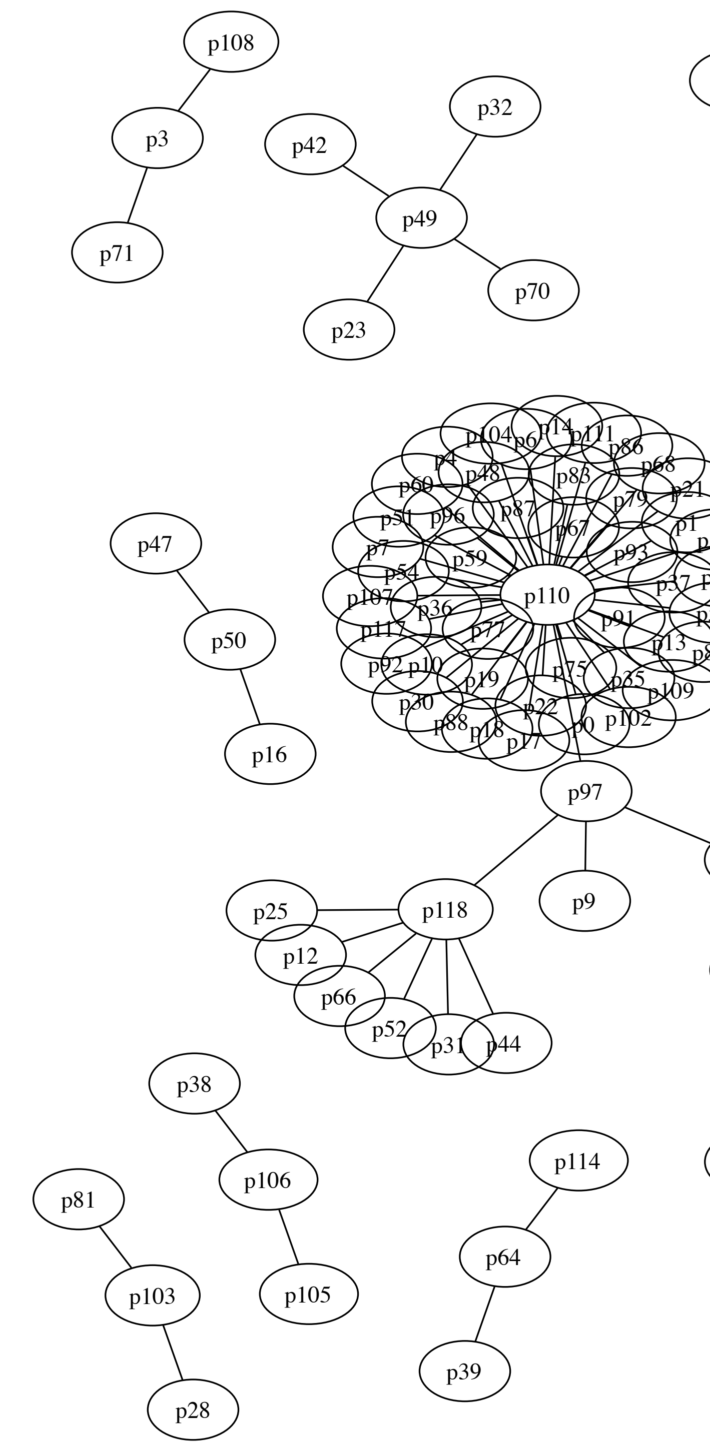

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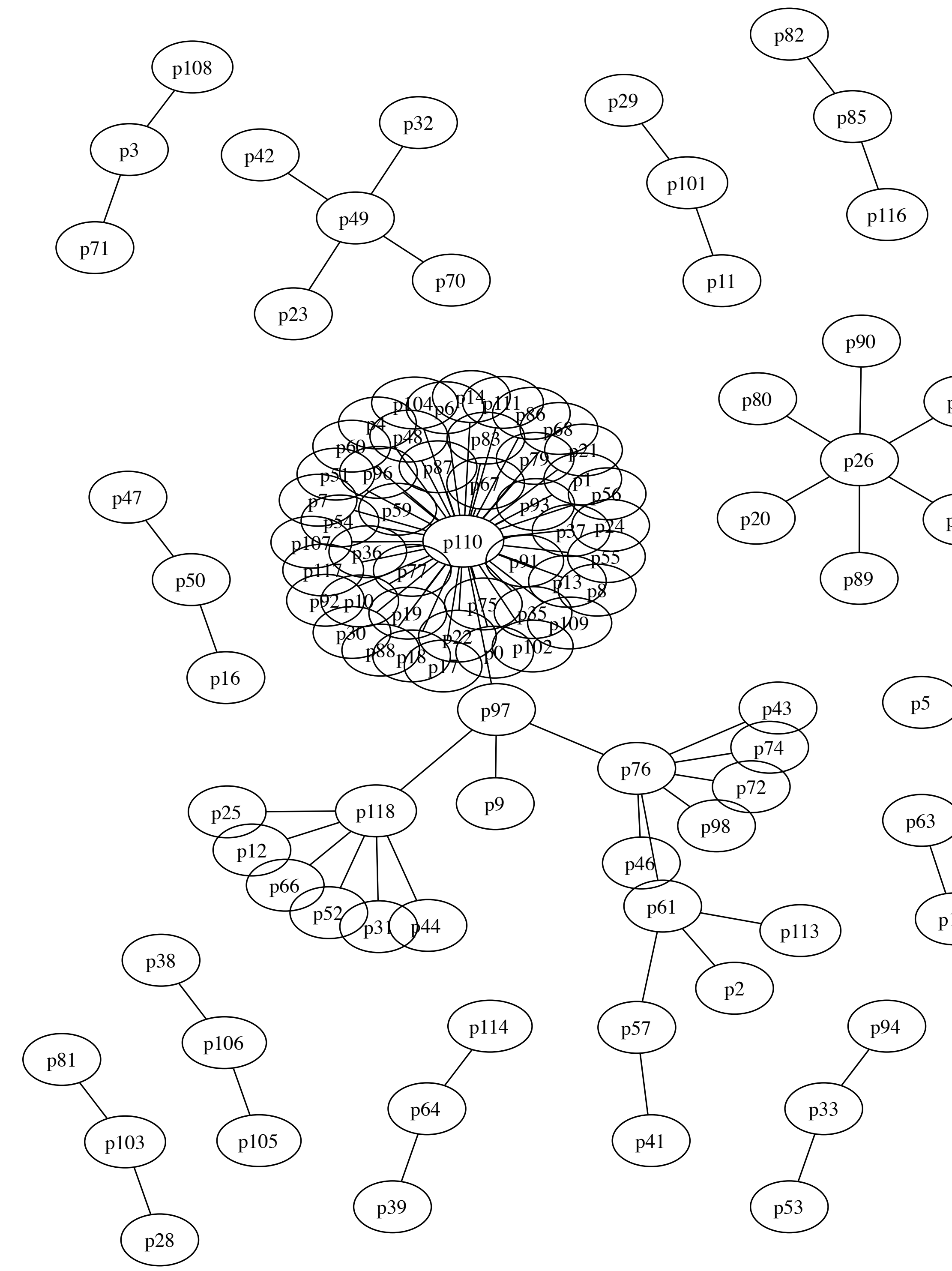

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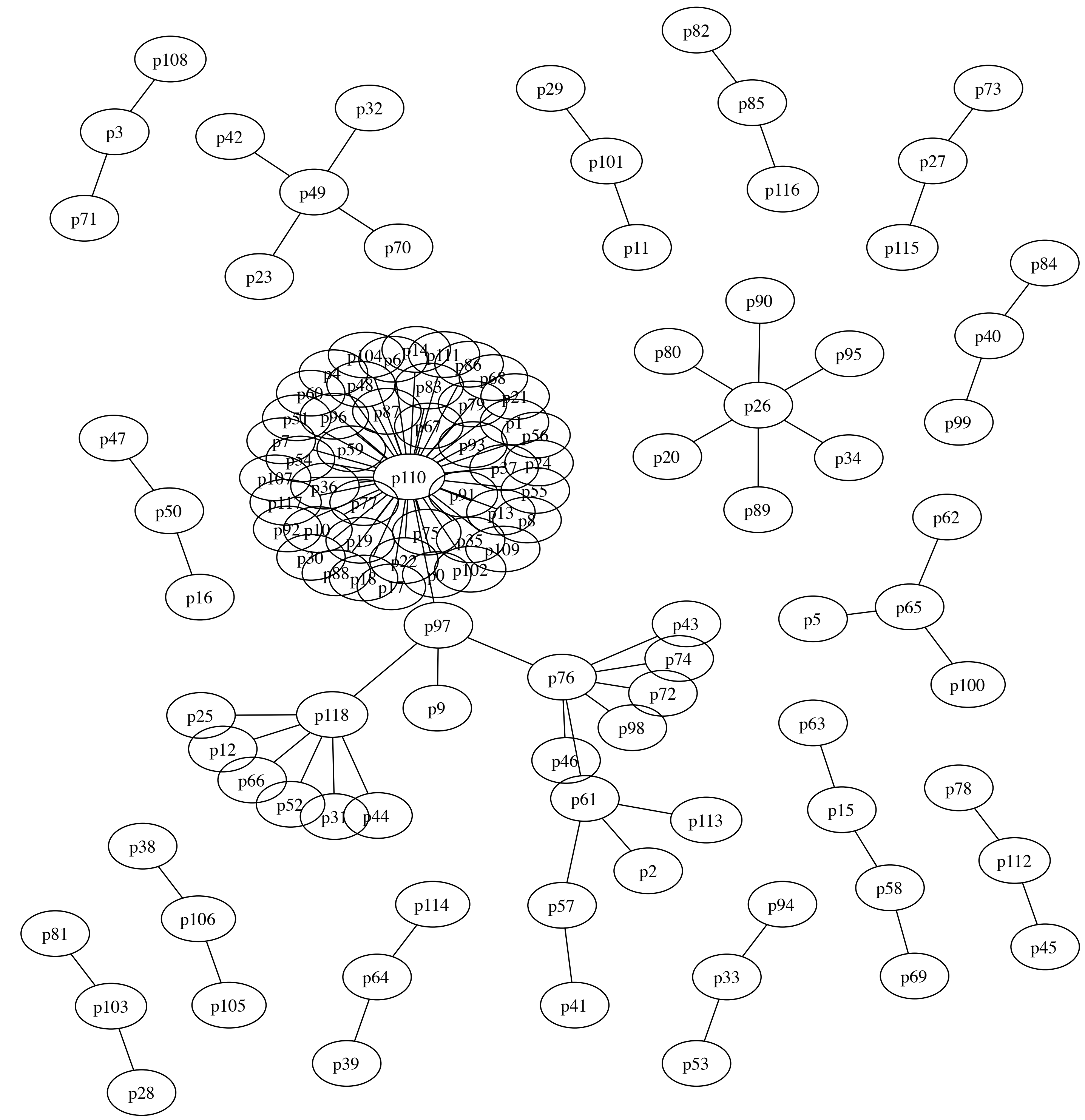

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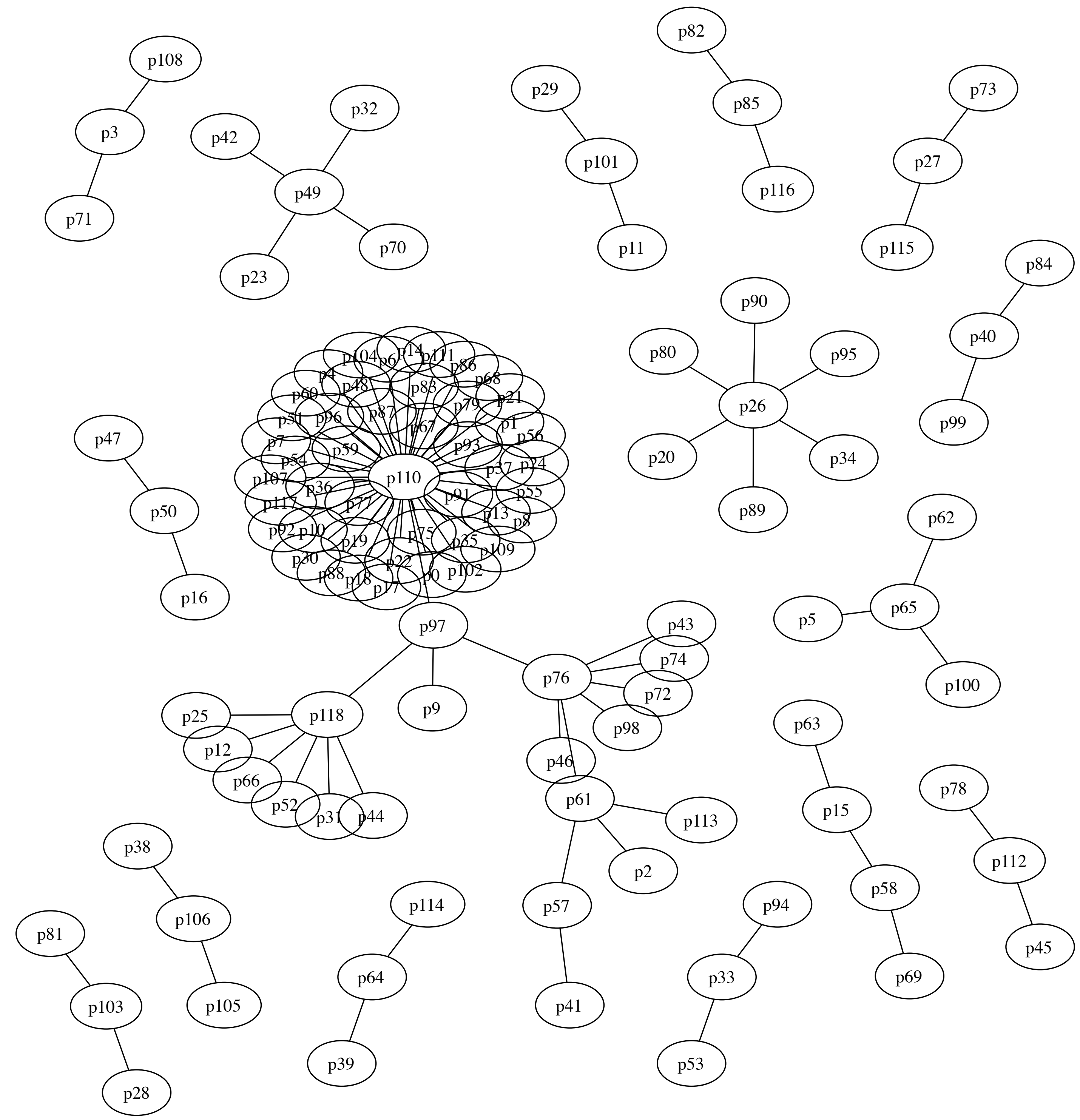

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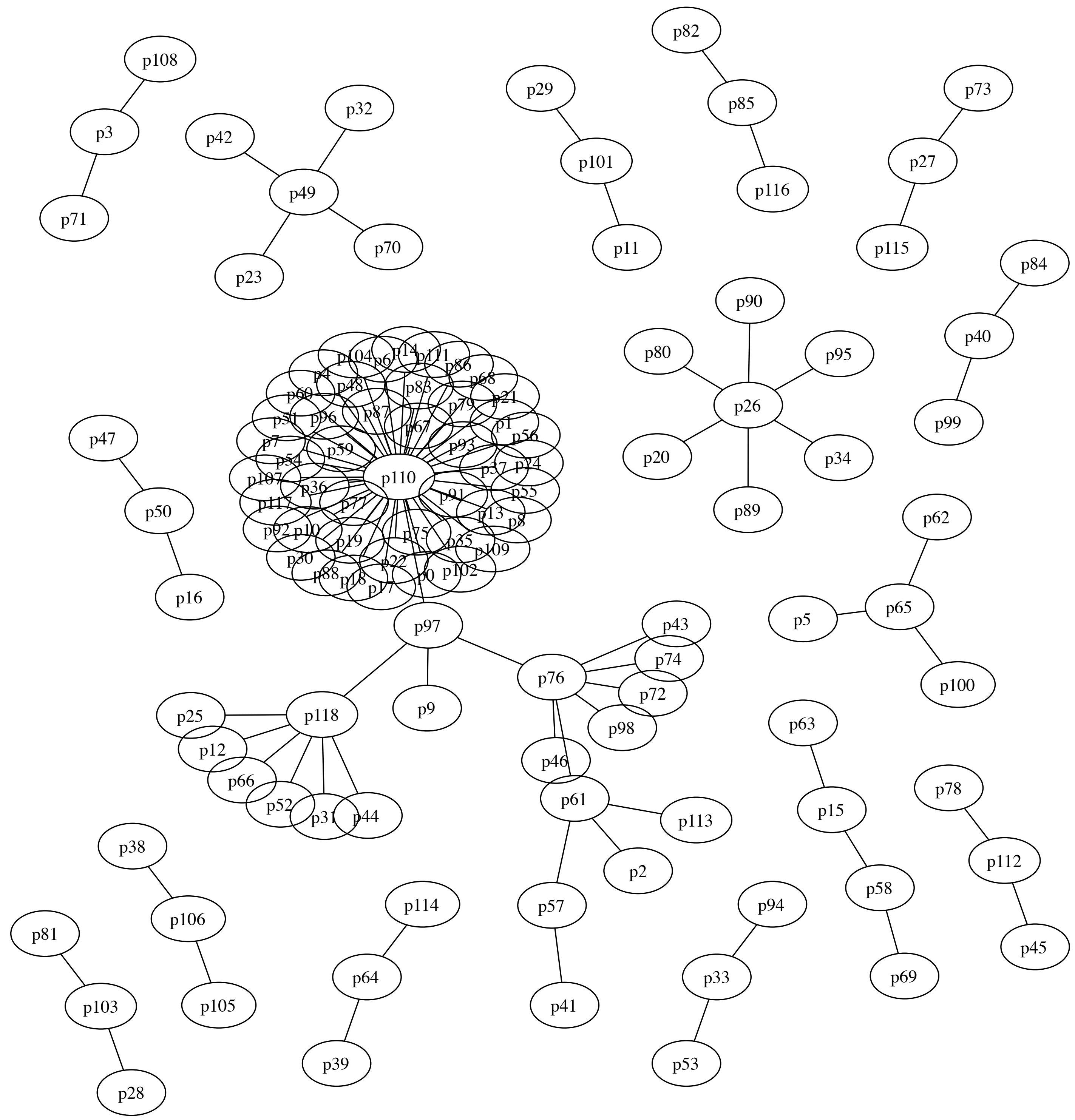

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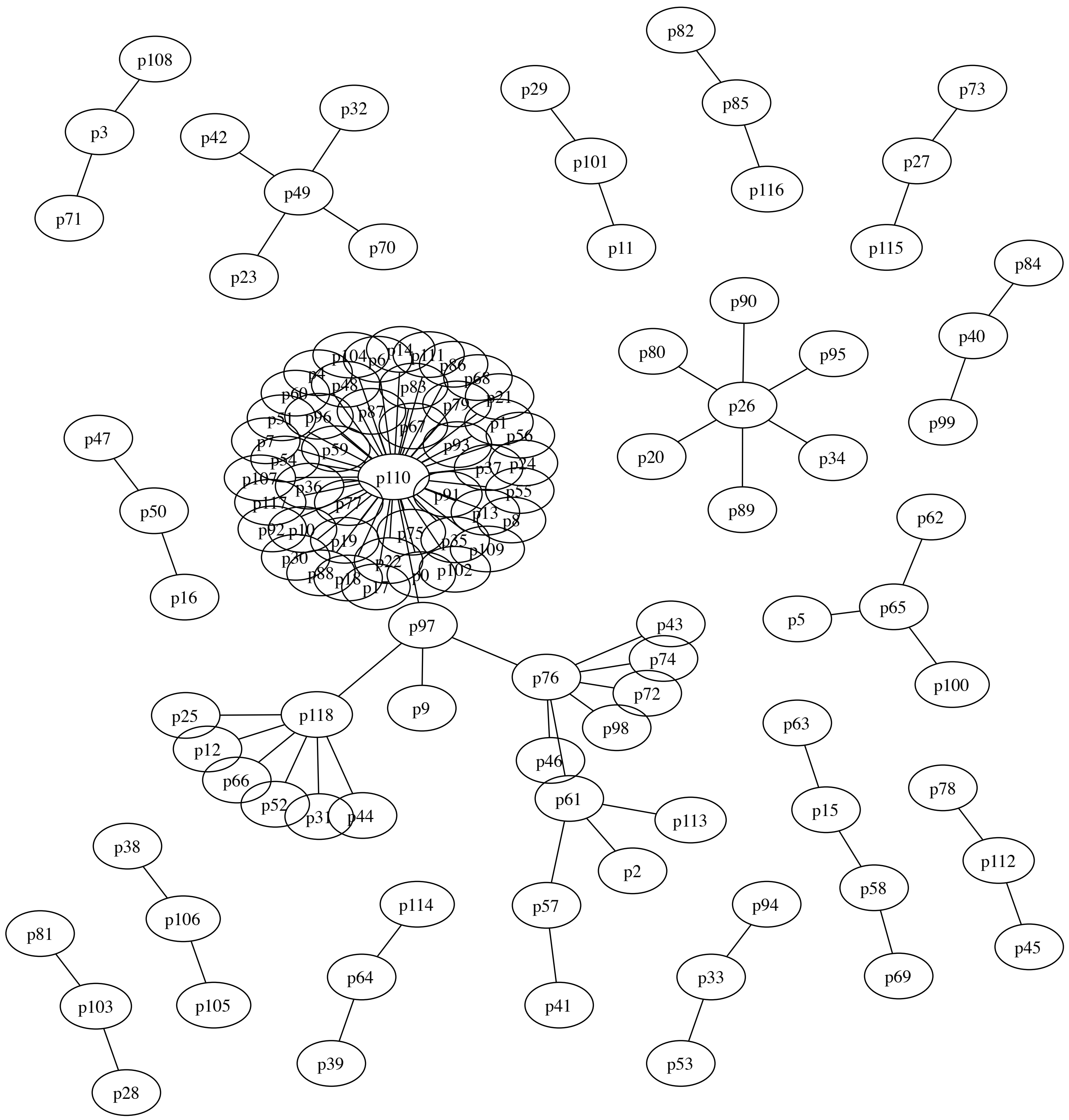

Prime p110 appears 46 time

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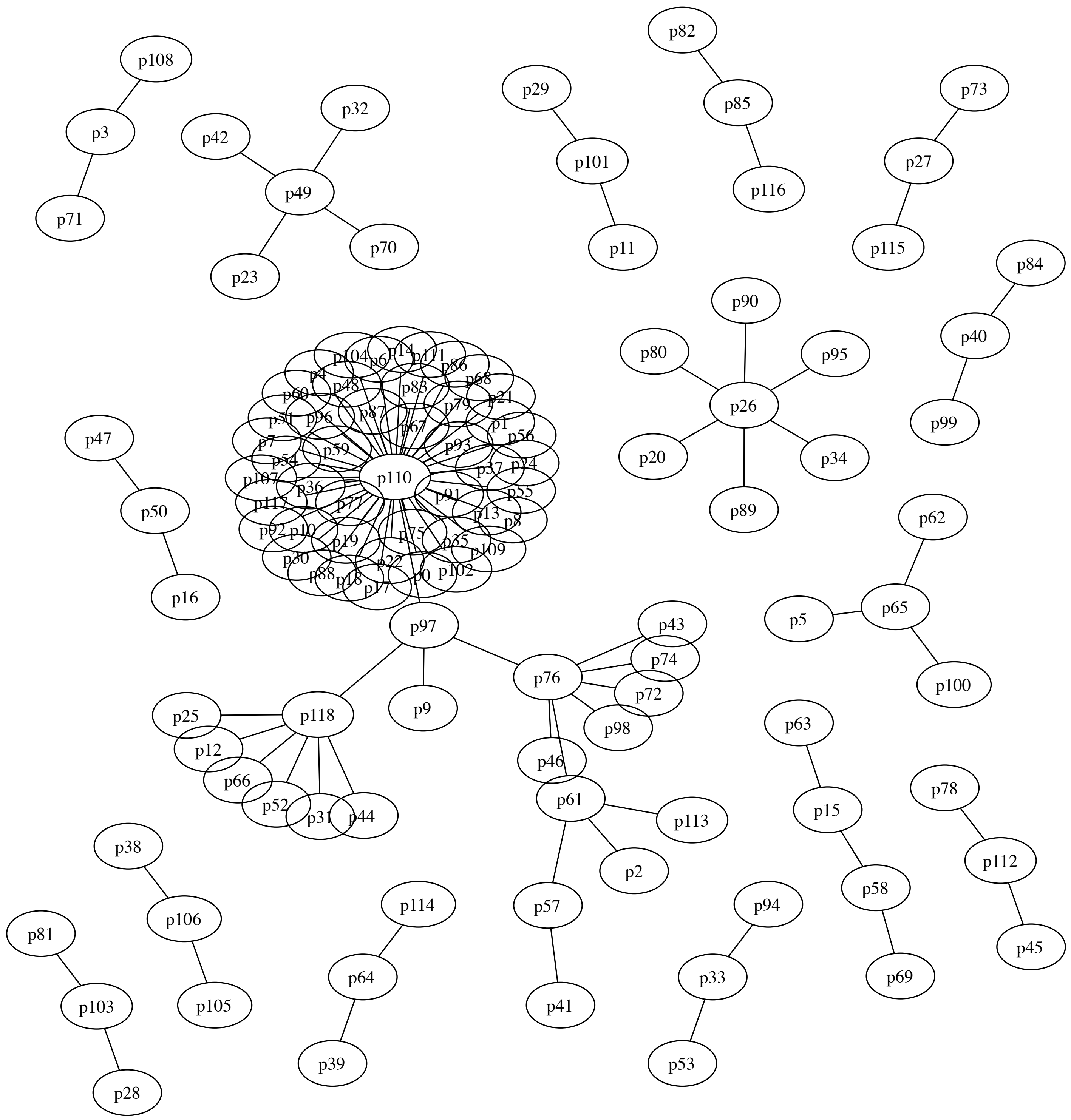

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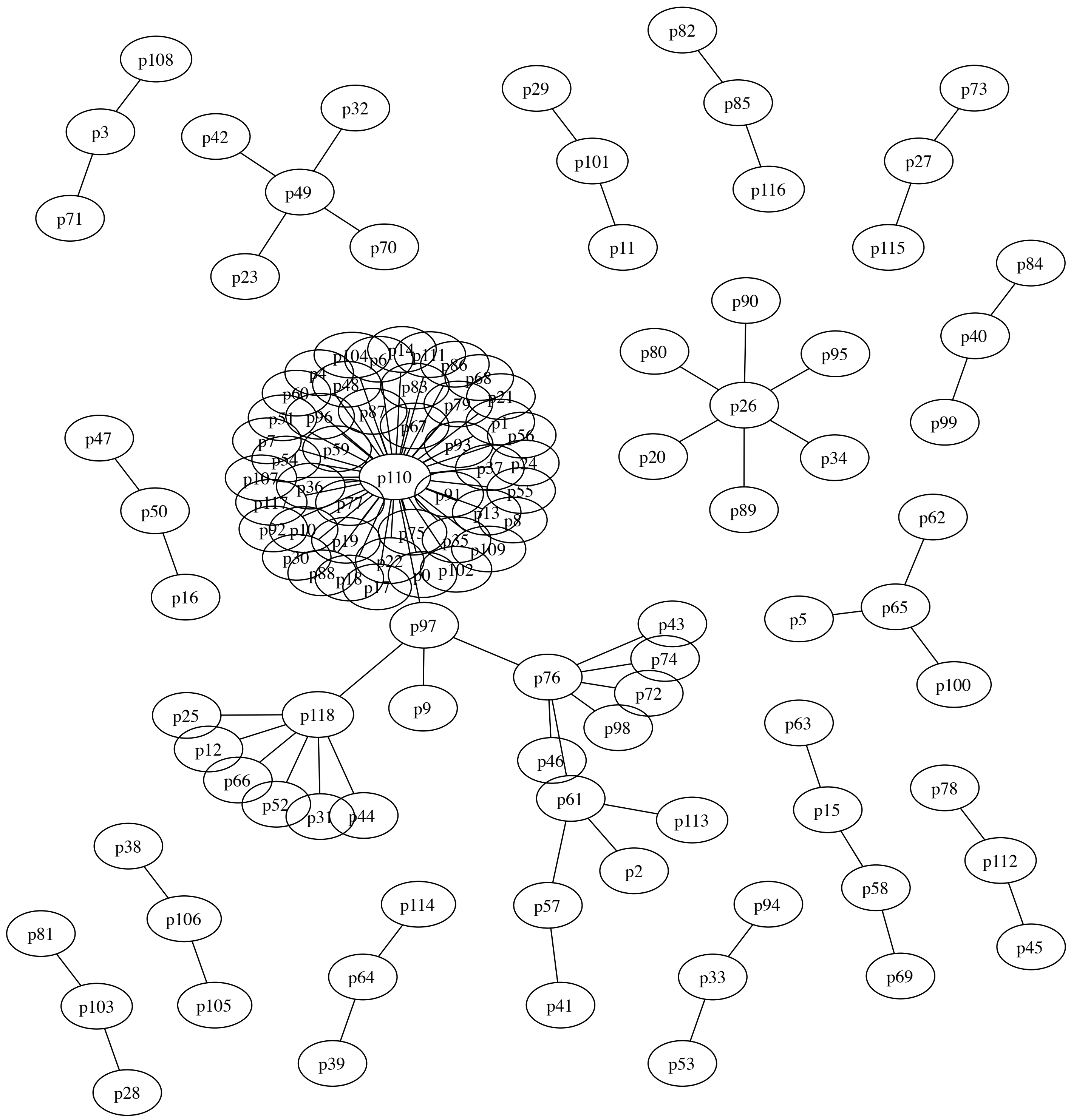

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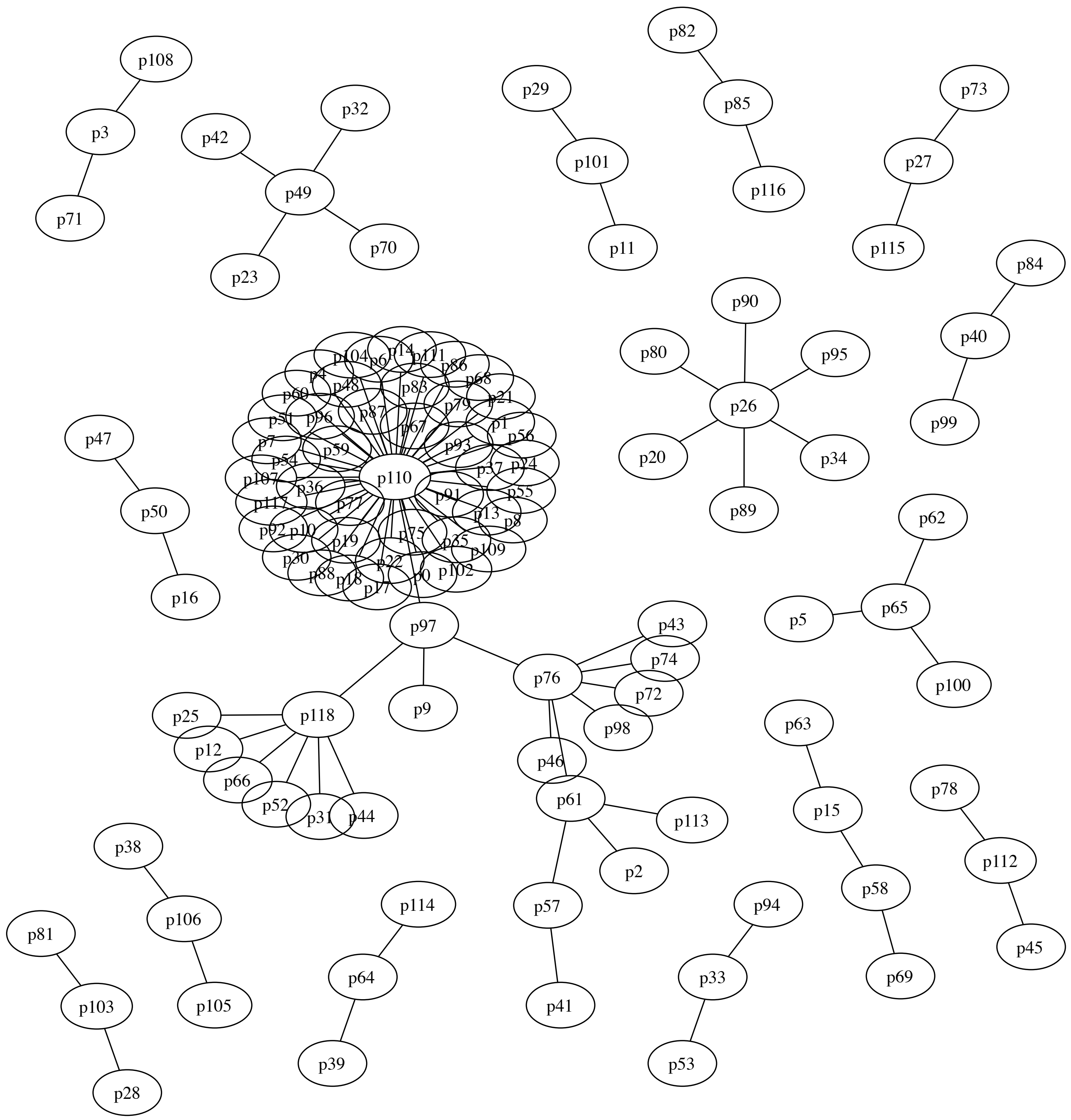

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Several other factors exhibit such a pattern.

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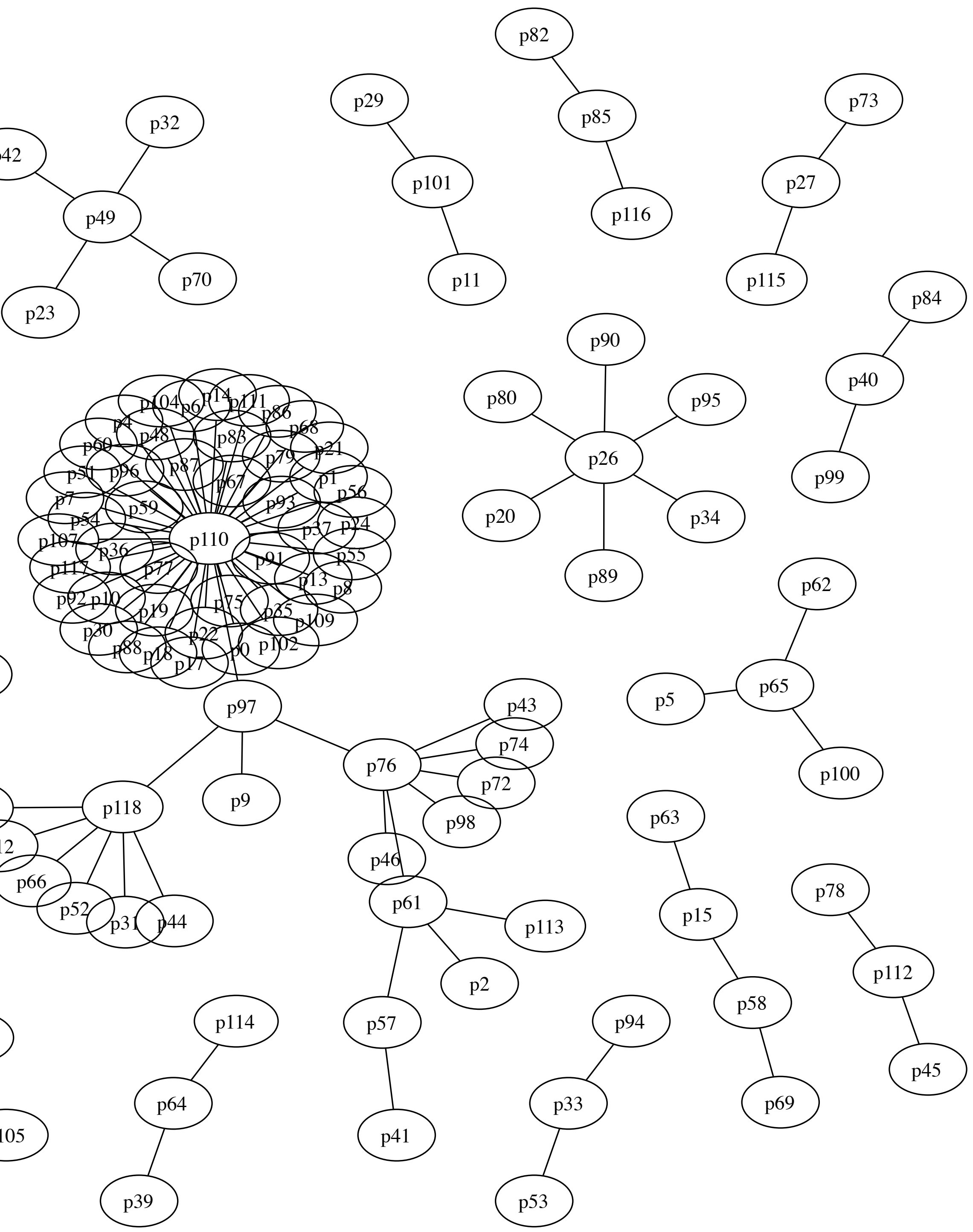

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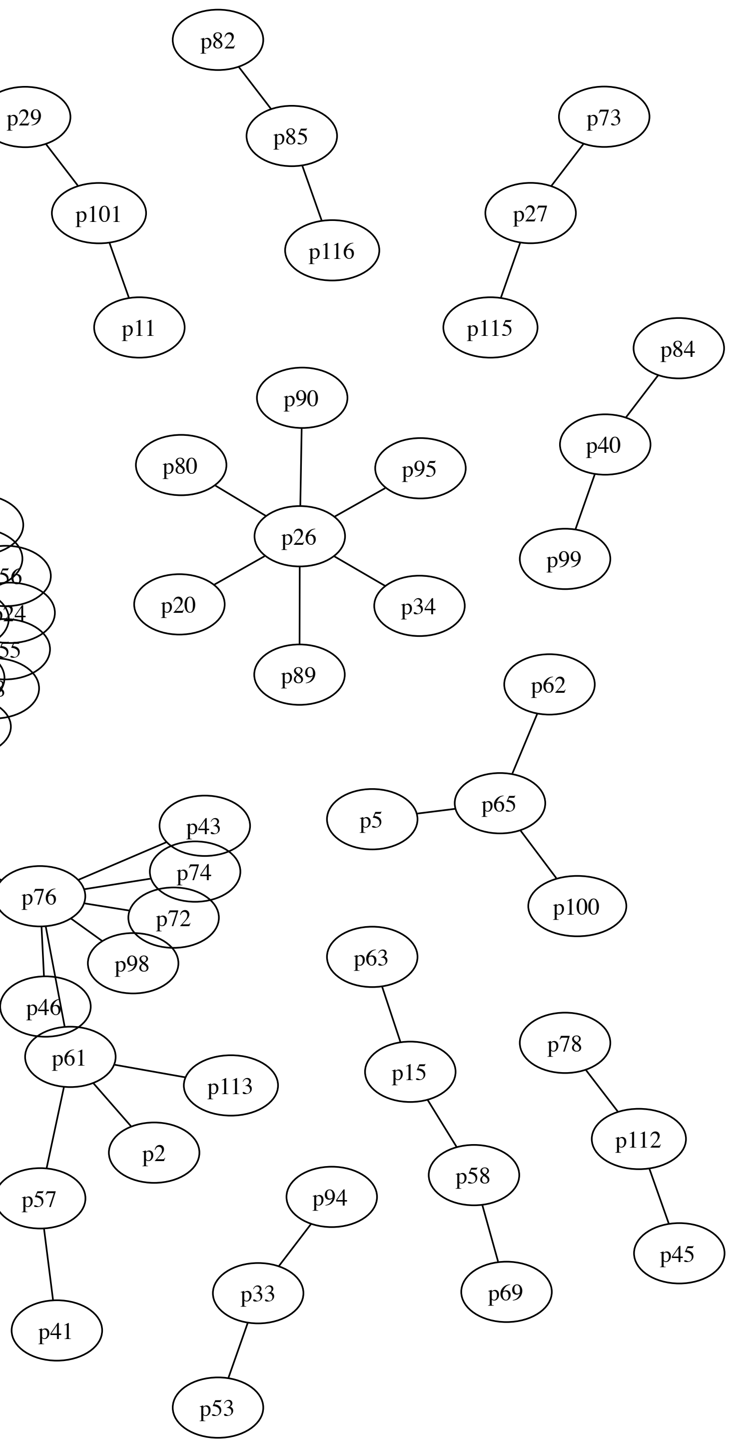

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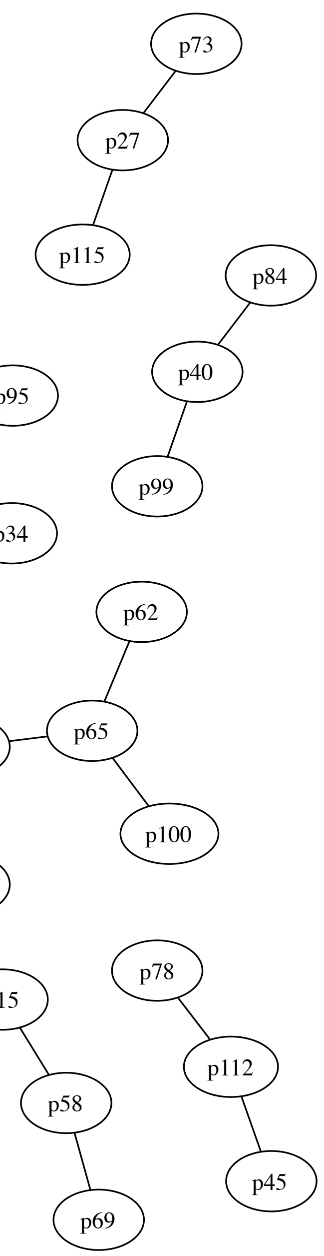

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Do this for any pattern:
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Factored 4 more keys using patterns of length 9.

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(with D.J. Bernstein, Y.-A. Chang, C.-M. Cheng, L.-P. Chou, N. Heninger, N. van Someren)
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$n \bmod 97 \in \{1, 35$
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Bad RSA randomness 2017

M. Nemec, M. Sys, P. Svend
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Bad RSA randomness 2017 – ROCA

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Coppersmith, Howgrave-Graham,
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$\log_2 L > 970 > 683 > 2048/3$.

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with r_i random and p' reconstructed using CRT.

Note: 2 and 3 are generator
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$2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 = 5760$ opti

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Note: 2 is not always a gene
this gives only
$2 \cdot 4 \cdot 3 \cdot 10 \cdot 12 = 2880$ opti

What went wrong here?

It would have been OK to choose
p' as
$p' \equiv 2^{r_1} \bmod 3$
$p' \equiv 3^{r_2} \bmod 5$
$p' \equiv 3^{r_3} \bmod 7$
$p' \equiv 2^{r_4} \bmod 11$
$p' \equiv 2^{r_5} \bmod 13$
with r_i random and p'
reconstructed using CRT.

Note: 2 and 3 are generators,
so this gives
$2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 = 5760$ options.

It would have OK'ish
but worse
to choose p' as
$p' \equiv 2^{r_1} \bmod 3$
$p' \equiv 2^{r_2} \bmod 5$
$p' \equiv 2^{r_3} \bmod 7$
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It is real

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$p' \equiv 547$

with r r

Note:

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Instead

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Note: 2 is not always a generator,

this gives only

$2 \cdot 4 \cdot 3 \cdot 10 \cdot 12 = 2880$ options.

It is really bad

to replace this by

exponentiation and

$p' \equiv 5477^r \bmod 3$

with r random.

Note:

The orders of 547

modulo 3,5,7,11, a

are 2,4,6,2, and 6,

are linked.

Instead of $2 \cdot 4 \cdot 6$

this gives lcm$\{2, 4$

options.

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It would have OK'ish
but worse
to choose p' as
$p' \equiv 2^{r_1} \bmod 3$
$p' \equiv 2^{r_2} \bmod 5$
$p' \equiv 2^{r_3} \bmod 7$
$p' \equiv 2^{r_4} \bmod 11$
$p' \equiv 2^{r_5} \bmod 13$
with r_i random and p'
reconstructed using CRT.

Note: 2 is not always a generator,
this gives only
$2 \cdot 4 \cdot 3 \cdot 10 \cdot 12 = 2880$ options.

It is really bad
to replace this by a single
exponentiation and choose p
$p' \equiv 5477^r \bmod 3 \cdot 5 \cdot 7 \cdot 11$
with r random.

Note:
The orders of 5477
modulo 3,5,7,11, and 13
are 2,4,6,2, and 6, but the p
are linked.

Instead of $2 \cdot 4 \cdot 6 \cdot 2 \cdot 6 =$
this gives $\text{lcm}\{2, 4, 6, 2, 6\} =$
options.

It would have OK'ish
but worse
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$p' \equiv 2^{r_1} \bmod 3$
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with r random.

Note:
The orders of 5477
modulo 3,5,7,11, and 13
are 2,4,6,2, and 6, but the powers
are linked.

Instead of $2 \cdot 4 \cdot 6 \cdot 2 \cdot 6 = 576$
this gives $\operatorname{lcm}\{2, 4, 6, 2, 6\} = 12$
options.