S-unit attacks

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(with lots of slides from Daniel J. Bernstein)

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KpqC

Post-quantum cryptography

Cryptography under the assumption that the attacker has a quantum computer.

- 1994: Shor's quantum algorithm. 1996: Grover's quantum algorithm.
  Many subsequent papers on quantum algorithms: see quantumalgorithmzoo.org.
- 2003: Daniel J. Bernstein introduces term Post-quantum cryptography.
- 2006: First International Workshop on Post-Quantum Cryptography. PQCrypto 2006, 2008, 2010, 2011, 2013, 2014, 2016, 2017, 2018, 2019, 2020, 2021, (soon) 2022.
- 2015: NIST hosts its first workshop on post-quantum cryptography.
- 2016: NIST announces a standardization project for post-quantum systems.
- 2017: Deadline for submissions to the NIST competition.
- 2019: Second round of NIST competition begins.
- 2020: Third round of NIST competition begins.
- ~~2021~~ 2022 ~~"not later than the end of March"~~:

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- 2020: Third round of NIST competition begins.
- ~~2021~~ 2022 ~~"not later than the end of March"~~: 05 Jul NIST announces first selections.
- 2022 $\to \infty$ NIST studies further systems.
- 2023/2024?: NIST issues post-quantum standards.

Major categories of public-key post-quantum systems

- **Code-based** encryption: McEliece cryptosystem has survived since 1978. Short ciphertexts and large public keys. Security relies on hardness of decoding error-correcting codes.
- **Hash-based** signatures: very solid security and small public keys. Require only a secure hash function (hard to find second preimages).
- **Isogeny-based** encryption: new kid on the block, promising short keys and ciphertexts and non-interactive key exchange. Security relies on hardness of finding isogenies between elliptic curves over finite fields.
- **Lattice-based** encryption and signatures: possibility for balanced sizes. Security relies on hardness of finding short vectors in some (typically special) lattice.
- **Multivariate-quadratic** signatures: short signatures and large public keys. Security relies on hardness of solving systems of multivariate equations over finite fields.

Warning: These are categories of mathematical problems;
individual systems may be totally insecure if the problem is not used correctly.

We have a good algorithmic abstraction of what a quantum computer can do,
but new systems need more analysis. Any extra structure offers more attack surface.

NIST's 5 July announcement

The winners:

- Kyber, a KEM based on structured lattices
- Dilithium, a signature scheme based on structured lattices
- Falcon, a signature scheme based on structured lattices
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Schemes advancing to round 4, so maybe more winners later:

- BIKE, a KEM based on codes
- Classic McEliece, a KEM based on codes
- HQC, a KEM based on codes
- SIKE, a KEM based on isogenies (now really badly broken, <1 month after NIST's announcement)

Lattice-based cryptography

1998 (ANTS-III) Hoffstein, Pipher, and Silverman introduce NTRU, working in ring $\mathbb{Z}[x]/(x^m - 1)$ (modulo q and modulo 3)

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2010 Lyubashevsky, Peikert, and Regev "introduce" Ring-LWE and prove "very strong hardness guarantees"

$$\begin{array}{c} \boxed{\begin{array}{c}\textbf{Assume} \text{ "worst-case problems on ideal lattices} \\ \text{are hard for polynomial-time quantum algorithms"}\end{array}} \\ \Downarrow \\ \boxed{\text{"the ring-LWE distribution is pseudorandom"}} \\ \Downarrow \\ \boxed{\begin{array}{c}\text{security for a "truly practical} \\ \text{lattice-based public-key cryptosystem"}\end{array}} \end{array}$$

Concrete parameters in cryptosystems are chosen assuming much more than polynomial hardness.

Typical structured lattices

NTRU uses $\mathbb{Z}[x]/(x^m - 1)$ for prime m.

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The winners all use 2-power cyclotomics:
Define $R = \mathbb{Z}[x]/(x^n + 1)$ for some $n \in \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \dots\}$.
From now on consider this case.

Ideal-SVP
Given a nonzero ideal $I \subseteq R$, find a "short" nonzero element $g \in I$.

Ideal I is given by basis $v_1, v_2, \dots, v_n \in R$ such that $I = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_n$.

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E.g. for $n = 4$

$v_1 = x^3 + 817$	$\longrightarrow$	$g = 2v_1 + 3v_2 - 5v_3 - 2v_4$
$v_2 = x^2 + 540$	this needs work	$= 2x^3 + 3x^2 - 5x + 1$
$v_3 = x + 247$		
$v_4 = 1009$		

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

817	0	0	1
540	0	1	0
247	1	0	0
1009	0	0	0

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Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 817 & 0 & 0 & 1 \\ 540 & 0 & 1 & 0 \\ 247 & 1 & 0 & 0 \\ 192 & 0 & 0 & -1 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 277 & 0 & -1 & 1 \\ 540 & 0 & 1 & 0 \\ 247 & 1 & 0 & 0 \\ 192 & 0 & 0 & -1 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$
\begin{array}{rrrr}
277 & 0 & -1 & 1 \\
263 & 0 & 2 & -1 \\
247 & 1 & 0 & 0 \\
192 & 0 & 0 & -1
\end{array}
$$

Doesn't look so hard ...

Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 263 & 0 & 2 & -1 \\ 247 & 1 & 0 & 0 \\ 192 & 0 & 0 & -1 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 247 & 1 & 0 & 0 \\ 192 & 0 & 0 & -1 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row
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$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 55 & 1 & 0 & 1 \\ 192 & 0 & 0 & -1 \end{array}$$

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$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 55 & 1 & 0 & 1 \\ 137 & -1 & 0 & -2 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{matrix} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 55 & 1 & 0 & 1 \\ 82 & -2 & 0 & -3 \end{matrix}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 55 & 1 & 0 & 1 \\ 27 & -3 & 0 & -4 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 28 & 4 & 0 & 5 \\ 27 & -3 & 0 & -4 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 1 & 7 & 0 & 9 \\ 27 & -3 & 0 & -4 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 1 & 7 & 0 & 9 \\ 11 & -2 & -2 & -3 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{matrix} 14 & 0 & -3 & 2 \\ 2 & -1 & 5 & -3 \\ 1 & 7 & 0 & 9 \\ 11 & -2 & -2 & -3 \end{matrix}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ 1 & 7 & 0 & 9 \\ 11 & -2 & -2 & -3 \end{array}$$

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$$\begin{array}{rrrr} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ -2 & 5 & 1 & 4 \\ 6 & -3 & -6 & -5 \end{array}$$

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Naive lattice-basis reduction: Reduce largest row
by subtracting closest multiple of another row.

$$\begin{matrix} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ -2 & 5 & 1 & 4 \\ 4 & 2 & -5 & -1 \end{matrix}$$

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Naive lattice-basis reduction: Reduce largest row
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Last row matches the $g = 2v_1 + 3v_2 - 5v_3 - 2v_4 = 2x^3 + 3x^2 - 5x + 1$ from above (up to sign).

But this doesn't reach "short" when n is large.

Lower bound on shortest nonzero element

Let $K = \mathbb{Q}(\zeta_{2n})$ and let $\iota_1, \iota_3, \ldots, \iota_{n-1}, \iota_{-1}, \ldots, \iota_{-(n-1)}$ be the embeddings of K into $\mathbb{C}$.
For $z \in \mathbb{C}$ let $|z| = \sqrt{z \cdot \bar{z}}$.

Minkowski embedding:
Apply $\left\{\iota_1, \ldots, \iota_{n-1}, \iota_{-1}, \ldots, \iota_{-(n-1)}\right\}$ to the nonzero ideal $I \subseteq R = \mathbb{Z}[x]/(x^n + 1)$.
Obtain an n-dim lattice of covolume $\sqrt{n^n} \cdot \#(R/I)$.

E.g., for $n = 4$ as above $1009 \mapsto (1009, 1009, 1009, 1009)$;
$x + 247 \mapsto (\zeta_8^1 + 247, \zeta_8^3 + 247, \zeta_8^{-3} + 247, \zeta_8^{-1} + 247)$;
$x^2 + 540 \mapsto (\zeta_8^2 + 540, \zeta_8^6 + 540, \zeta_8^{-6} + 540, \zeta_8^{-2} + 540)$;
$x^3 + 817 \mapsto (\zeta_8^3 + 817, \zeta_8^9 + 817, \zeta_8^{-9} + 817, \zeta_8^{-3} + 817)$;
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$x^3 + 817 \mapsto (\zeta_8^3 + 817, \zeta_8^9 + 817, \zeta_8^{-9} + 817, \zeta_8^{-3} + 817)$;
$I \hookrightarrow\!\!\!\!\rightarrow$ 4-dim lattice of covolume $4^{4/2} \cdot 1009 \approx 11.27^4$;

Use this to bound length of $g \in I - \{0\}$ with $\prod_\iota |\iota(g)| = \#(R/g) \geq \#(R/I)$ so
$||g||_2 = \sqrt{\sum_\iota |\iota(g)|^2} \geq \sqrt{n}(\prod_\iota |\iota(g)|)^{1/n} \geq \sqrt{n}\#(R/I)^{1/n} = (\text{covol}\, I)^{1/n}$.

In our example $g = 2x^3 + 3x^2 - 5x + 1 \mapsto$
$(2\zeta_8^3 + 3\zeta_8^2 - 5\zeta_8 + 1, 2\zeta_8^9 + 3\zeta_8^6 - 5\zeta_8^3 + 1, 2\zeta_8^{-9} + 3\zeta_8^{-6} - 5\zeta_8^{-3} + 1, 2\zeta_8^{-3} + 3\zeta_8^{-2} - 5\zeta_8^{-1} + 1)$
$||g||_2 = \sqrt{4}\sqrt{2^2 + 3^2 + 5^2 + 1} \approx 12.49 > 11.27$.

Upper bound on shortest nonzero element

1889 Minkowski “geometry of numbers” implies

$$||g||_2 \le 2(n/2)!^{1/n}\pi^{-1/2}(\mathrm{covol}\ I)^{1/n}$$

for some $g \in I - \{0\}$, i.e., some nonzero $g \in I$ has

$$\eta = \frac{||g||_2}{(\mathrm{covol}\ I)^{1/n}} \le 2(n/2)!^{1/n}\pi^{-1/2},$$

where η is called the “Hermite factor”.

E.g. $n = 4$: $\eta \le 1.35$. $n = 512$: $\eta \le 11.03$.
Have $2(n/2)!^{1/n}\pi^{-1/2} \approx \sqrt{2n/e\pi}$ for large n.

This shows that very short elements exist.
But can we find them?

Performance of known algorithms

Algorithm input: nonzero ideal $I \subseteq R = \mathbb{Z}[x]/(x^n + 1)$.
Output: nonzero $g = g_0 + \cdots + g_{n-1}x^{n-1} \in I$ with $(g_0^2 + \cdots + g_{n-1}^2)^{1/2} = \eta \cdot (\#(R/I))^{1/n}$.

Algorithms using only additive structure of I:

- LLL (fast): $\eta^{1/n} \approx 1.022$.
- BKZ-80 (not hard): $\eta^{1/n} \approx 1.010$.
- BKZ-160 (public attack): $\eta^{1/n} \approx 1.007$.
- BKZ-300 (large-scale attack): $\eta^{1/n} \approx 1.005$.

BKZ-β repeatedly computes a shortest basis in a lattice of dimension β.
Quality and cost increae with β.

These algorithms work for arbitrary lattices.
Can we do better using ideal structure?

Notation for infinite places of $K = \mathbb{Q}[x]/(x^n + 1)$

Define $\zeta_m = \exp(2\pi i/m) \in \mathbb{C}$ for nonzero $m \in \mathbb{Z}$.

For any $c \in 1 + 2\mathbb{Z}$ have $(\zeta_{2n}^c)^n + 1 = 0$ so there is a unique ring morphism $\iota_c : K \to \mathbb{C}$ taking x to ζ_{2n}^c.

All roots of $x^n + 1$ in $\mathbb{C}$: $\zeta_{2n}^1, \dots, \zeta_{2n}^{n-1}, \zeta_{2n}^{-(n-1)}, \dots, \zeta_{2n}^{-1}$.
All $\iota : K \to \mathbb{C}$: $\iota_1, \dots, \iota_{n-1}, \iota_{-(n-1)}, \dots, \iota_{-1}$.

Define $|g|_c = |\iota_c(g)|^2 = \iota_c(g)\iota_{-c}(g)$.
The maps $g \mapsto |g|_c$ are the **infinite places** of K.
All infinite places: $g \mapsto |g|_1, g \mapsto |g|_3, \dots, g \mapsto |g|_{n-1}$.
Same as: $g \mapsto |g|_{-1}, g \mapsto |g|_{-3}, \dots, g \mapsto |g|_{-n-1}$.

$$\sum_{c \in \{1,3,\dots,n-1\}} |g_0 + \cdots + g_{n-1}x^{n-1}|_c = \frac{n}{2}(g_0^2 + \cdots + g_{n-1}^2).$$

Notation for finite places of $K = \mathbb{Q}[x]/(x^n + 1)$

Nonzero ideals of R factor into prime ideals.

For each nonzero prime ideal P of R, define

$$|g|_P = \#(R/P)^{-\mathrm{ord}_P g}.$$

"Norm of P" is $\#(R/P)$.
The maps $g \mapsto |g|_P$ are the **finite places** of K.

For each prime number p:
Factor $x^n + 1$ in $\mathbb{F}_p[x]$ to see the prime ideals of R containing p.

E.g. $p = 2$: Prime ideal $2R + (x + 1)R = (x + 1)R$.

E.g. "unramified degree-1 primes":
$p \in 1 + 2n\mathbb{Z} \Rightarrow$ exactly n nth roots $r_1, \ldots, r_n$ of -1 in $\mathbb{F}_p$.
$x^n + 1 = (x - r_1)(x - r_2)\ldots(x - r_n)$ in $\mathbb{F}_p[x]$.
Prime ideals $pR + (x - r_1)R, \ldots, pR + (x - r_n)R$.

Notation for places $g \mapsto |g|_v$ for, e.g., $n = 4$, $R = \mathbb{Z}[x]/(x^4+1)$

$$g = g_0 + g_1 x + g_2 x^2 + g_3 x^3, \qquad \zeta_8 = \exp(2\pi i/8):$$

$$\iota_{-1}(g) = g_0 + g_1\zeta_8^{-1} + g_2\zeta_8^{-2} + g_3\zeta_8^{-3};$$

$$\iota_1(g) = g_0 + g_1\zeta_8 + g_2\zeta_8^{2} + g_3\zeta_8^{3}; \qquad |g|_1 = |\iota_1(g)|^2.$$

$$\iota_{-3}(g) = g_0 + g_1\zeta_8^{-3} + g_2\zeta_8^{-6} + g_3\zeta_8^{-9};$$

$$\iota_3(g) = g_0 + g_1\zeta_8^{3} + g_2\zeta_8^{6} + g_3\zeta_8^{9}; \qquad |g|_3 = |\iota_3(g)|^2.$$

$P_{17,2} = 17R + (x-2)R$: $\qquad |g|_{17,2} = 17^{-\mathrm{ord}_{P_{17,2}} g}$.

$P_{17,8} = 17R + (x-8)R$: $\qquad |g|_{17,8} = 17^{-\mathrm{ord}_{P_{17,8}} g}$.

$P_{17,-8} = 17R + (x+8)R$: $\qquad |g|_{17,-8} = 17^{-\mathrm{ord}_{P_{17,-8}} g}$.

$P_{17,-2} = 17R + (x+2)R$: $\qquad |g|_{17,-2} = 17^{-\mathrm{ord}_{P_{17,-2}} g}$.

$P_{41,3} = 41R + (x-3)R$: $\qquad |g|_{41,3} = 41^{-\mathrm{ord}_{P_{41,3}} g}$.

etc.

S-units of $K = \mathbb{Q}[x]/(x^n + 1)$

Assume $\infty \subseteq S \subseteq \{\text{places of } K\}$.
Useful special case: S has all primes $\leq y$ for some y.
[Warning: Often people rename $S - \infty$ as S.]

$$
\begin{aligned}
g \in K^* \text{ is an } \boldsymbol{S}\textbf{-unit} \quad &\Leftrightarrow \quad gR = \prod_{P \in S} P^{e_P} \text{ for some } e_P \\
&\Leftrightarrow \quad |g|_v = 1 \text{ for all } v \in \{\text{places of } K\} - S \\
&\Leftrightarrow \quad \text{the vector } v \mapsto \log |g|_v \text{ is 0 outside } S.
\end{aligned}
$$

***S*-unit lattice:** set of such vectors $v \mapsto \log |g|_v$.

E.g. Temporarily allowing $n = 1$, $K = \mathbb{Q}$:
$\{\{\infty, 2, 3\}\text{-units in } \mathbb{Q}\} = \pm 2^{\mathbb{Z}} 3^{\mathbb{Z}}$. ("3-smooth".)
Lattice: $(\log 2, -\log 2, 0)\mathbb{Z} + (\log 3, 0, -\log 3)\mathbb{Z}$.

Special case: unit attacks

0. Define $S = \infty$. $\{\infty\text{-units of } K\} = \{\text{units of } R\} = R^*$.
1. Input a nonzero ideal I of R.
2. Find a generator of I: some g with $gR = I$.
3. Find a unit u "close to g".
4. Output g/u.

This assumes R^* is known and I is principal.

Quality of the output:
How small is g/u compared to I?
Most cryptosystems require approx SVP to be hard.

History: 2014 Bernstein: this is "reasonably well known among computational algebraic number theorists" and is a threat to lattice-based cryptography.
2014 Campbell–Groves–Shepherd: exploit cyclotomic units to break a lattice-based system from 2009 Gentry. Assume finding g with quantum algorithm.
2015 Cramer–Ducas–Peikert–Regev: asymptotic analysis of 2014 algorithm.

S-unit attacks

0. Choose a finite set S of places including ∞.
1. Input a nonzero ideal I of R.
2. Find an S-generator of I: some g with $gR = I \prod_{P \in S} P^{e_P}$.
3. Find an S-unit u "close to g/I". This is an S-unit-lattice close-vector problem.
4. Output g/u.

Step 2 has a poly-time quantum algorithm from 2016 Biasse–Song,
building on unit-group algorithm from 2014 Eisenträger–Hallgren–Kitaev–Song.
Also has non-quantum algorithms running in subexponential time,
assuming standard heuristics; for analysis and speedups see 2014 Biasse–Fieker.

Critical for Step 3 speed: constructing short vectors in the S-unit lattice.

History: 2015 Bernstein: apply unit attacks to close principal multiple of I.
2016 Bernstein: S-unit attacks.
2017 Cramer–Ducas–Wesolowski: use cyclotomic structure in finding close principal multiples;
more analysis in 2019 Ducas–Plançon–Wesolowski.
2019 Pellet-Mary–Hanrot–Stehlé: first analysis of S-unit attacks.
See also 2020 Bernard–Roux-Langlois, 2021 Bernard–Lesavourey–Nguyen–Roux-Langlois.

“Cyclotomic units” in $R = \mathbb{Z}[x]/(x^n + 1)$

$\pm 1, \pm x, \pm x^2, \ldots, \pm x^{n-1} = \mp 1/x$ are units.

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This is a unit since $(1 - x)/(1 - x^3) = (1 - x^{2n^2+1})/(1 - x^3) \in R$.

For $c \in 1 + 2\mathbb{Z}$: R has automorphism $\sigma_c : x \mapsto x^c$.
$\sigma_c(1 + x + x^2) = 1 + x^c + x^{2c}$ is a unit.
Useful to symmetrize: define $u_c = 1 + x^c + x^{-c}$.

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$\pm 1, \pm x, \pm x^2, \ldots, \pm x^{n-1} = \mp 1/x$ are units.

$(1 - x^3)/(1 - x) = 1 + x + x^2 \in R$.
This is a unit since $(1 - x)/(1 - x^3) = (1 - x^{2n^2+1})/(1 - x^3) \in R$.

For $c \in 1 + 2\mathbb{Z}$: R has automorphism $\sigma_c : x \mapsto x^c$.
$\sigma_c(1 + x + x^2) = 1 + x^c + x^{2c}$ is a unit.
Useful to symmetrize: define $u_c = 1 + x^c + x^{-c}$.

$x^{\mathbb{Z}} \prod_c u_c^{\mathbb{Z}}$ has finite index in R^*. Index is called h^+.
Assume $h^+ = 1$. Proven, assuming GRH, for $n \in \{2, 4, 8, \ldots, 256\}$; see 2014 Miller.
Heuristics say true for all powers of 2; see 2004 Buhler–Pomerance–Robertson, 2015 Miller.

Unit lattice for $n = 8$

$|u_1|_1 = |1 + \zeta_{16} + \zeta_{16}^{-1}|^2 \approx \exp 2.093.$
$|u_1|_3 = |1 + \zeta_{16}^3 + \zeta_{16}^{-3}|^2 \approx \exp 1.137.$
$|u_1|_5 = |1 + \zeta_{16}^5 + \zeta_{16}^{-5}|^2 \approx \exp -2.899.$
$|u_1|_7 = |1 + \zeta_{16}^7 + \zeta_{16}^{-7}|^2 \approx \exp -0.330.$

Define

$$\operatorname{Log}_\infty f = (\log|f|_1, \log|f|_3, \log|f|_5, \log|f|_7).$$

$\operatorname{Log}_\infty u_1 \approx (2.093, 1.137, -2.899, -0.330).$
$\operatorname{Log}_\infty u_3 \approx (1.137, -0.330, 2.093, -2.899).$
$\operatorname{Log}_\infty u_5 \approx (-2.899, 2.093, -0.330, 1.137).$
$\operatorname{Log}_\infty u_7 \approx (-0.330, -2.899, 1.137, 2.093).$

$\operatorname{Log}_\infty R^*$ is lattice of dim $n/2 - 1 = 3$ in hyperplane

$$\left\{(\ell_1, \ell_3, \ell_5, \ell_7) \in \mathbb{R}^4 : \ell_1 + \ell_3 + \ell_5 + \ell_7 = 0\right\}.$$

Short lattice basis: $\operatorname{Log}_\infty u_1$, $\operatorname{Log}_\infty u_3$, $\operatorname{Log}_\infty u_5$.

Reducing modulo units

Assume I is principal.
Start with generator $g = g_0 + g_1 x + \cdots + g_{n-1} x^{n-1}$ of I.
Compute $\operatorname{Log}_\infty g = (\log |g|_1, \log |g|_3, \ldots, \log |g|_{n-1})$.

Replacing g with gu replaces $|g|_c$ with $|g|_c |u|_c$.
Easy to track $||g||_2^2 = \sum_c |g|_c = (n/2)(g_0^2 + \cdots + g_{n-1}^2)$.

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Try to reduce $\text{Log}_\infty g$ modulo unit lattice:
Adjust $\text{Log}_\infty g$ by subtracting vectors from $\text{Log}_\infty(R^*)$.

Actually, precompute some combinations of basis vectors
and subtract closest vector within this set;
repeat several times; keep smallest $g_0^2 + \cdots + g_{n-1}^2$.

Note that unit hyperplane is orthogonal to norm:
$\#(R/I) = \#(R/g) = \prod_c |g|_c = \exp \sum_c \log |g|_c$.

Experiments for small n

Geometric average of $\eta^{1/n}$ over 100000 experiments:

n	Model	Attack	Tweak	Shortest
4	1.01516	1.01518	1.01518	1.01518
8	1.01968	1.01972	1.01696	1.01696
16	1.01861	1.01860	1.01628	1.01627

"Shortest": Take I, find a shortest nonzero vector g,
output $\eta = (g_0^2 + \cdots + g_{n-1}^2)^{1/2}/\#(R/I)^{1/n}$.
[*Assuming* BKZ-n software produces shortest nonzero vector.]

"Attack": Same I, find a generator, reduce mod unit lattice $\to g$,
output $(g_0^2 + \cdots + g_{n-1}^2)^{1/2}/\#(R/I)^{1/n}$.

"Model": Take a hyperplane point, reduce mod unit lattice $\to \mathrm{Log}_\infty g$,
output $(g_0^2 + \cdots + g_{n-1}^2)^{1/2}$.

"Tweak": Multiply by $x+1$, reduce, repeat for $I, (x+1)I, (x+1)^2 I, (x+1)^3 I, (x+1)^4 I, \ldots$.
Often $(x+1)^e g$ is closer to unit lattice than g.
(This is including a finite place of norm 2 in S.)

Nice S-units for cyclotomics (as in this talk)

Can use Gauss sums and Jacobi sums.
For details and more credits see 2021 talk given by Bernstein at SIAM-AG.

For each prime number $p \in 1 + 2n\mathbb{Z}$, and each group morphism $\chi : \mathbb{F}_p^* \to \zeta_{2n}^{\mathbb{Z}}$, define

$$\text{Gauss}\Sigma_p(\chi) = \sum_{a \in \mathbb{F}_p^*} \chi(a)\zeta_p^a.$$

Then $\text{Gauss}\Sigma_p(\chi)$ is an S-unit for $S = \infty \cup p$.

E.g. $n = 16$, $\zeta_{2n} = \zeta_{32}$, $p = 97 \in 1 + 2n\mathbb{Z}$:
There is a morphism $\chi : \mathbb{F}_{97}^* \to \zeta_{32}^{\mathbb{Z}}$ with $\chi(5) = \zeta_{32}$.
$\text{Gauss}\Sigma_p(\chi) = \zeta_{32}^0\zeta_{97}^1 + \zeta_{32}^1\zeta_{97}^5 + \zeta_{32}^2\zeta_{97}^{25} + \cdots$.
$\text{Gauss}\Sigma_p(\chi^2) = \zeta_{32}^0\zeta_{97}^1 + \zeta_{32}^2\zeta_{97}^5 + \zeta_{32}^4\zeta_{97}^{25} + \cdots$.

Stickelberger and augmented Stickelberger lattices used in 2019 Ducas–Plançon–Wesolowski are exponent vectors in factorizations of (some) ratios of Gauss sums.

Traditional method to find S-units: filtering

Take random small element $u \in R$: e.g. $u = x^{31} - x^{41} + x^{59} + x^{26} - x^{53}$.

1. Does $\#(R/u)$ factor into primes $\le y$?
   Needs fast computation of norms and factorization.
   Lots of algorithmic speedups.
2. Is u an S-unit for $S = \infty \cup \{P : \#(R/P) \le y\}$?

Small primes $\Rightarrow$ fast non-quantum factorization.
[Helpful speedups: almost always $\#(R/P) \in 1 + 2n\mathbb{Z}$. Batch factorization.]

Standard heuristics $\Rightarrow$ $y^{2+o(1)}$ choices of u include $y^{1+o(1)}$ S-units, spanning all S-units, for

- appropriate $n^{1/2+o(1)}$ choice for $\log y$,
- appropriate $n^{1/2+o(1)}$ choice for $\sum_i u_i^2$.

Total time $\exp(n^{1/2+o(1)})$.

Can tricks from NFS on extensions be applied to reach $1/3 + o(1)$?

Automorphisms and subrings

Apply each σ_c to quickly amplify each u found into, typically, n independent S-units.

What if u is invariant under (say) two σ_c?

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Apply each σ_c to quickly amplify each u found into, typically, n independent S-units.

What if u is invariant under (say) two σ_c? Great!
Start with u from proper subrings. Makes $\#(R/u)$ much more likely to factor into small primes.

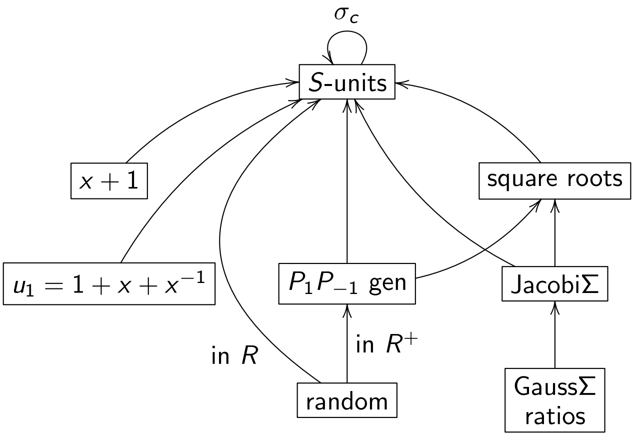
Examples of useful subrings of $R = \mathbb{Z}[x]/(x^n + 1)$:

- $\mathbb{Z}[x^2]/(x^n + 1) = \{u \in R : \sigma_{n+1}(u) = u\}$.
- $R^+ = \{u \in R : \sigma_{-1}(u) = u\}$.

Also use subrings to speed up $\#(R/u)$ computation: see
https://s-unit.attacks.cr.yp.to/norms.html.

Some rings (but not power-of-2 cyclotomics) have so many subrings that no other techniques are needed: see 2014 Bernstein, 2017 Bauch–Bernstein–de Valence–Lange–van Vredendaal, 2018 Biasse-van Vredendaal, 2020 Lesavourey–Plantard–Susilo, 2020 Biasse–Fieker–Hofmann–Page.

Overview: Constructing small S-units



Conjectured scalability: $\exp(n^{1/2+o(1)})$

Simple algorithm variant, skipping many speedups:

Take traditional $\log y \in n^{1/2+o(1)}$.
Take $S = \infty \cup \{P : \#(R/P) \le y\}$.
Precompute

$$\left\{S\text{-unit } u \in R\text{: } \sum_i u_i^2 \le n^{1/2+o(1)}\right\}.$$

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To randomize, multiply I by some random primes in S. Can repeat $y^{O(1)}$ times.

Compute S-generator g of I (quantum or classical).

Clear denominators: Multiply by generators of $P_c P_{-c}$ (this assumes $h^+ = 1$)
$\Rightarrow$ element of I that S-generates I.

Replace g with gu/v having log vector closest to I;
repeat until stable $\Rightarrow$ short element of I.

Heuristics $\Rightarrow \eta \le n^{1/2+o(1)}$, time $\exp(n^{1/2+o(1)})$.
"Vector within ε of shortest in subexponential time."

Compare to typical cryptographic assumption: $\eta \le n^{2+o(1)}$ is hard to reach.

Non-randomness of S-unit lattices

Number of points of a lattice L in a big ball $B \approx \dfrac{\operatorname{vol} B}{\operatorname{covol} L}$.

For almost all lattices L (1956 Rogers, ..., 2019 Strömbergsson–Södergren):
If $\operatorname{vol} B = \operatorname{covol} L$ then length of shortest nonzero vector in $L \approx$ radius of B.

2016 Laarhoven: analogous heuristics for effectiveness of reduction via subtracting off short vectors from database. 2019 Pellet-Mary–Hanrot–Stehlé, 2021 Ducas–Pellet-Mary: Apply these heuristics to S-unit lattices $\Rightarrow$ very small chance that previous slide works.

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But all of these heuristics provably fail for the lattice $\mathbb{Z}^d$.
Are these accurate for S-unit lattices?

2021 Bernstein–Lange "Non-randomness of S-unit lattices":
The standard length/reduction heuristics provably fail for S-unit lattices for (1) $n = 1$, any S; (2) each n as S grows (roughly what the previous slide uses); (3) minimal S, any n.
See `https://s-unit.attacks.cr.yp.to/spherical.html`.

Evidence for the conjecture

For traditional $\log y \in n^{1/2+o(1)}$, time budget $\exp(n^{1/2+o(1)})$:
Standard smoothness heuristics $\Rightarrow$ find short S-units spanning the S-unit lattice, as in 2014 Biasse–Fieker; and find S-generator of I.

Various quantifications of the behavior of S-unit lattices are much closer to $\mathbb{Z}^d$ than to random lattices.
Model reduction as $\mathbb{Z}^d$ reduction $\Rightarrow$ find short S-generator of I.

Full attack software now available: https://s-unit.attacks.cr.yp.to/filtered.html.
Numerical experiments are consistent with the heuristics.

Ongoing work: attack speedups; more precise S-unit models and predictions; more numerical evidence for comparison to the models; other fast S-unit constructions, exploiting more cyclotomic structure.