Elliptic-curve cryptography

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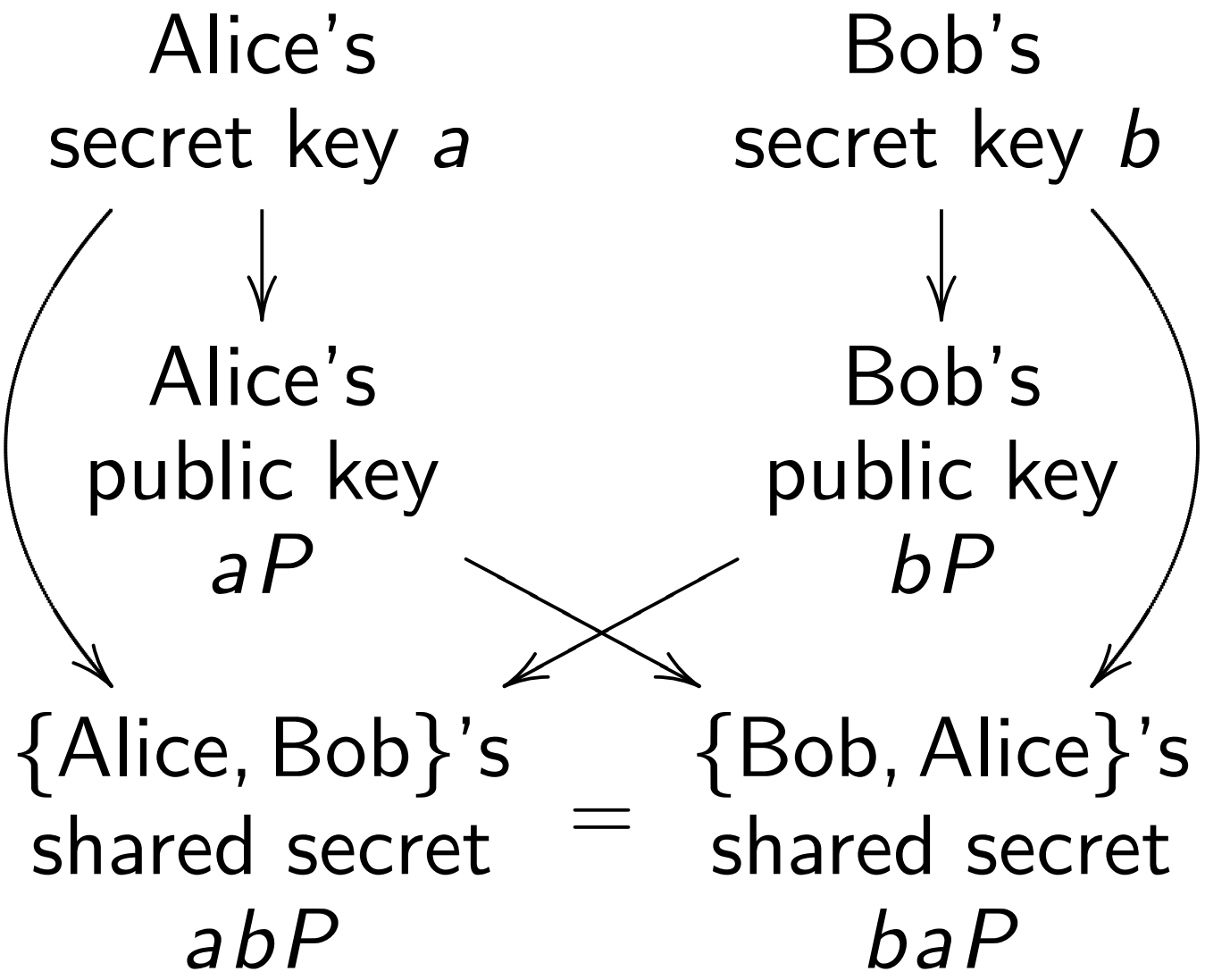
13 November 2020

with some slides by
Daniel J. Bernstein

Diffie-Hellman key exchange

Pick some *generator* P,
i.e. some group element
(using additive notation here).



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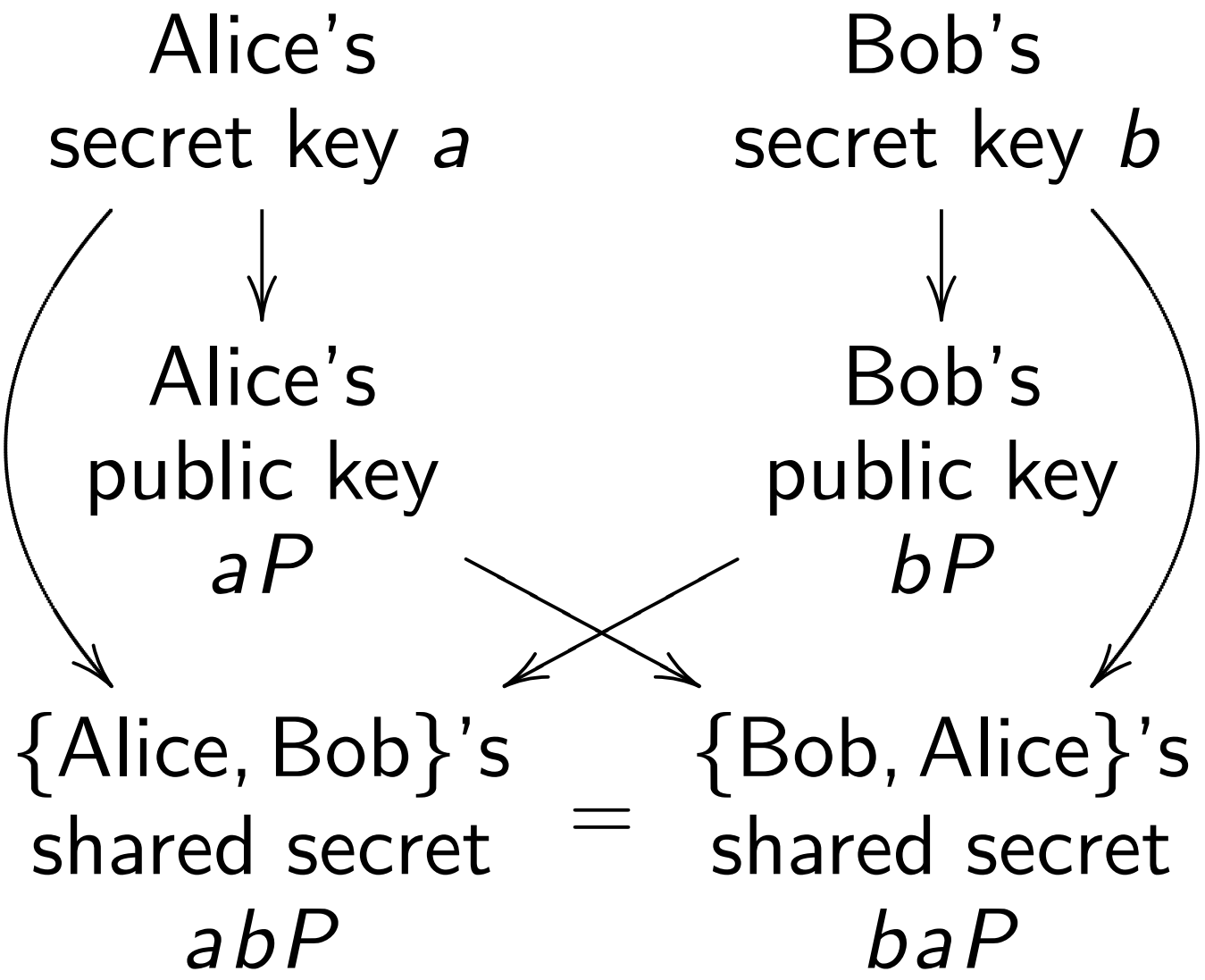
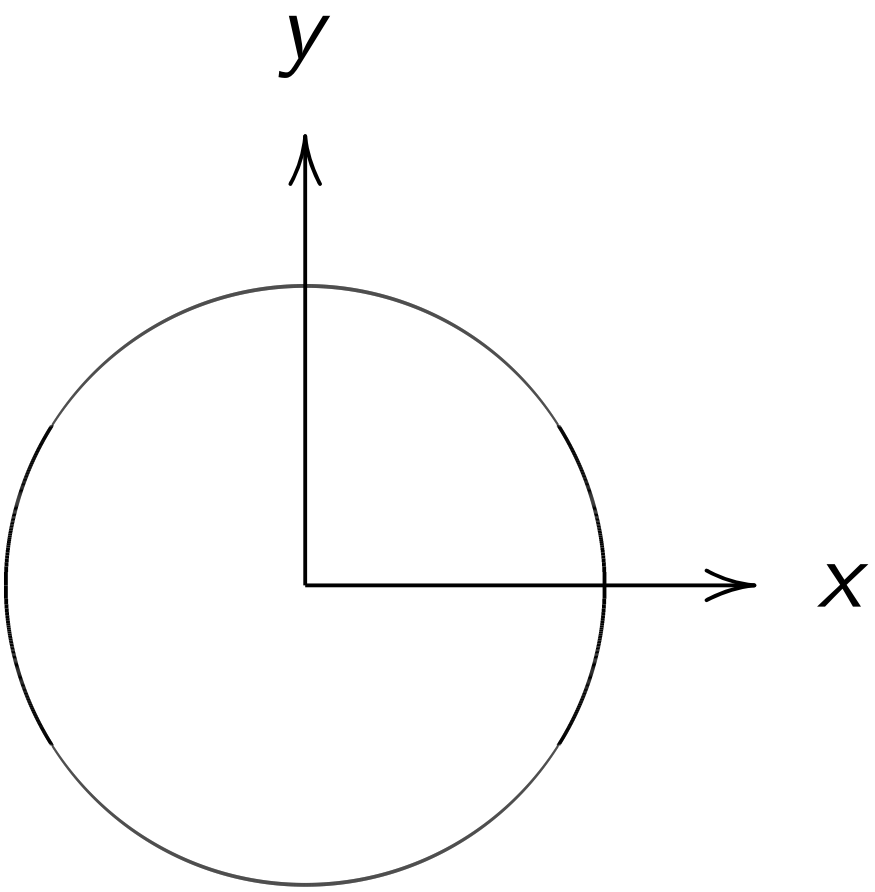


What does P look like &
how to compute $P + Q$?

The clock



This is the curve $x^2 + y^2 = 1$.

Warning:

This is *not* an elliptic curve.

"Elliptic curve" $\neq$ "ellipse."

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$(\sqrt{1/2}, \sqrt{1/2}) =$ “1:30”.

$(3/5, 4/5)$. $(-3/5, 4/5)$.

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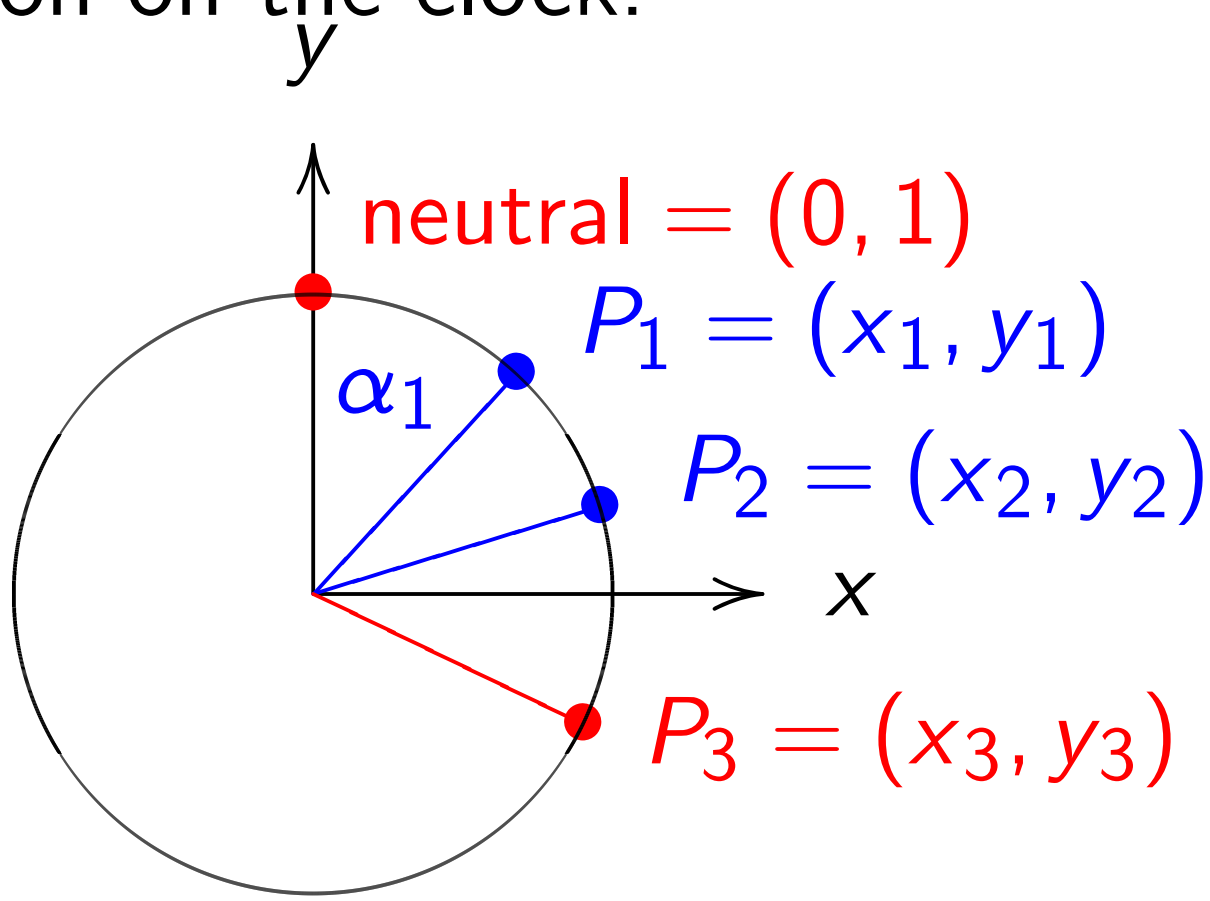
Many more.

Addition on the clock:



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$x = \sin\alpha, \quad y = \cos\alpha.$

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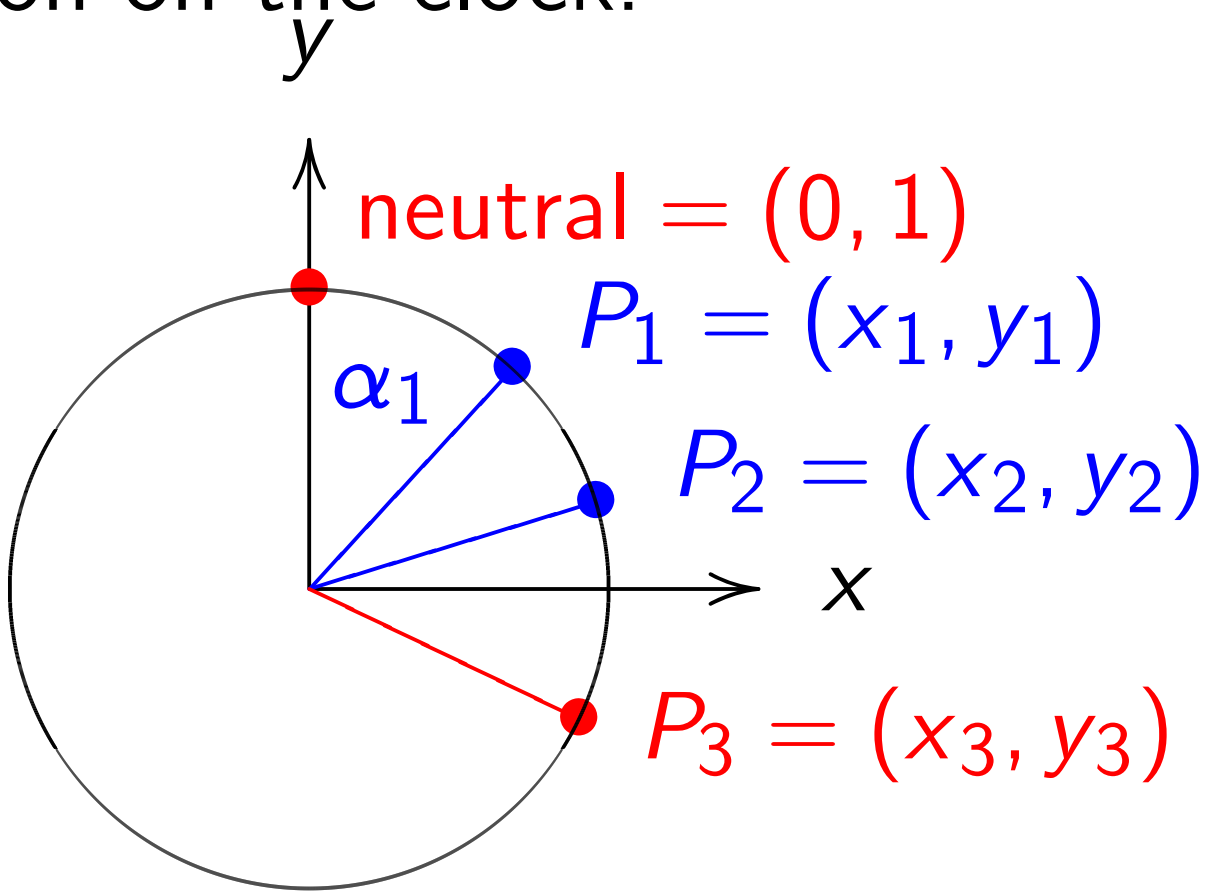

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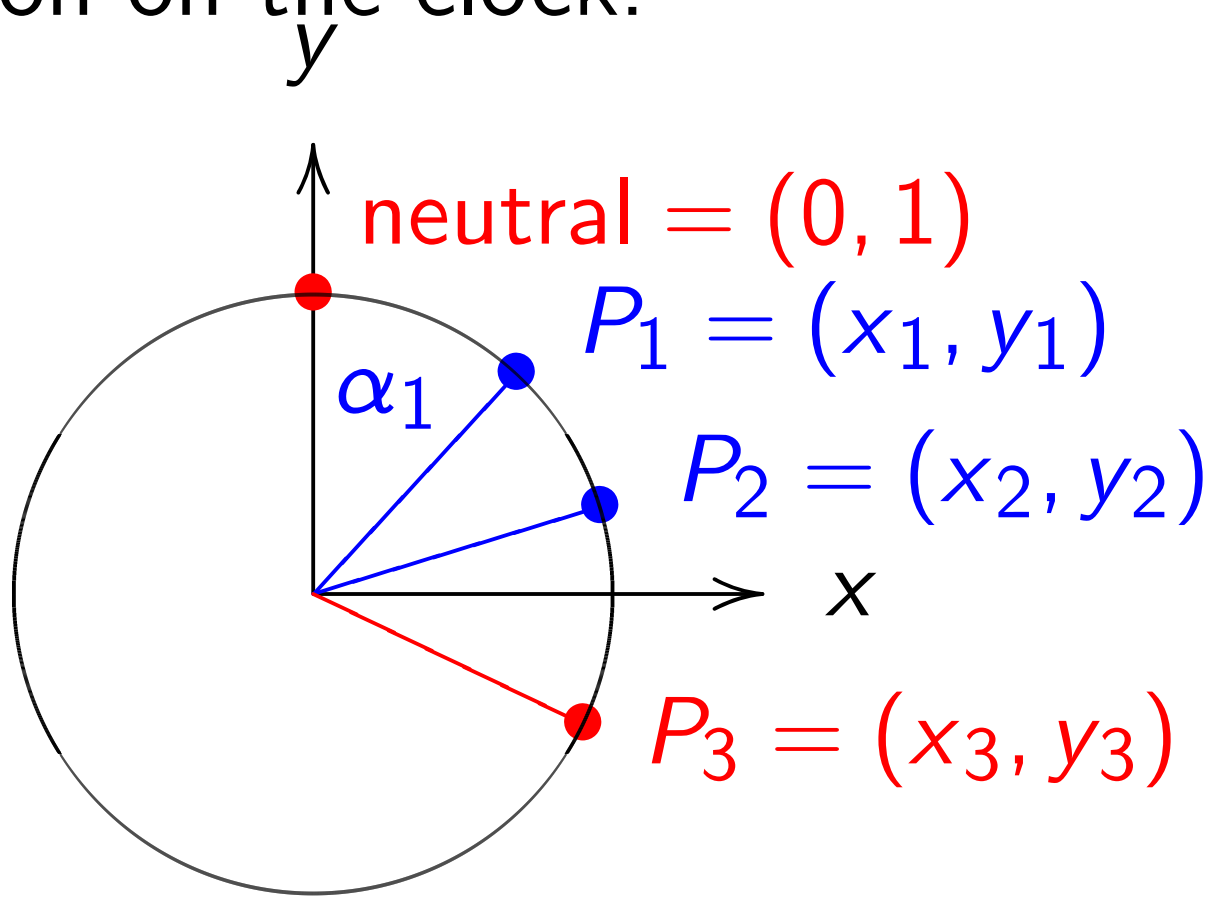

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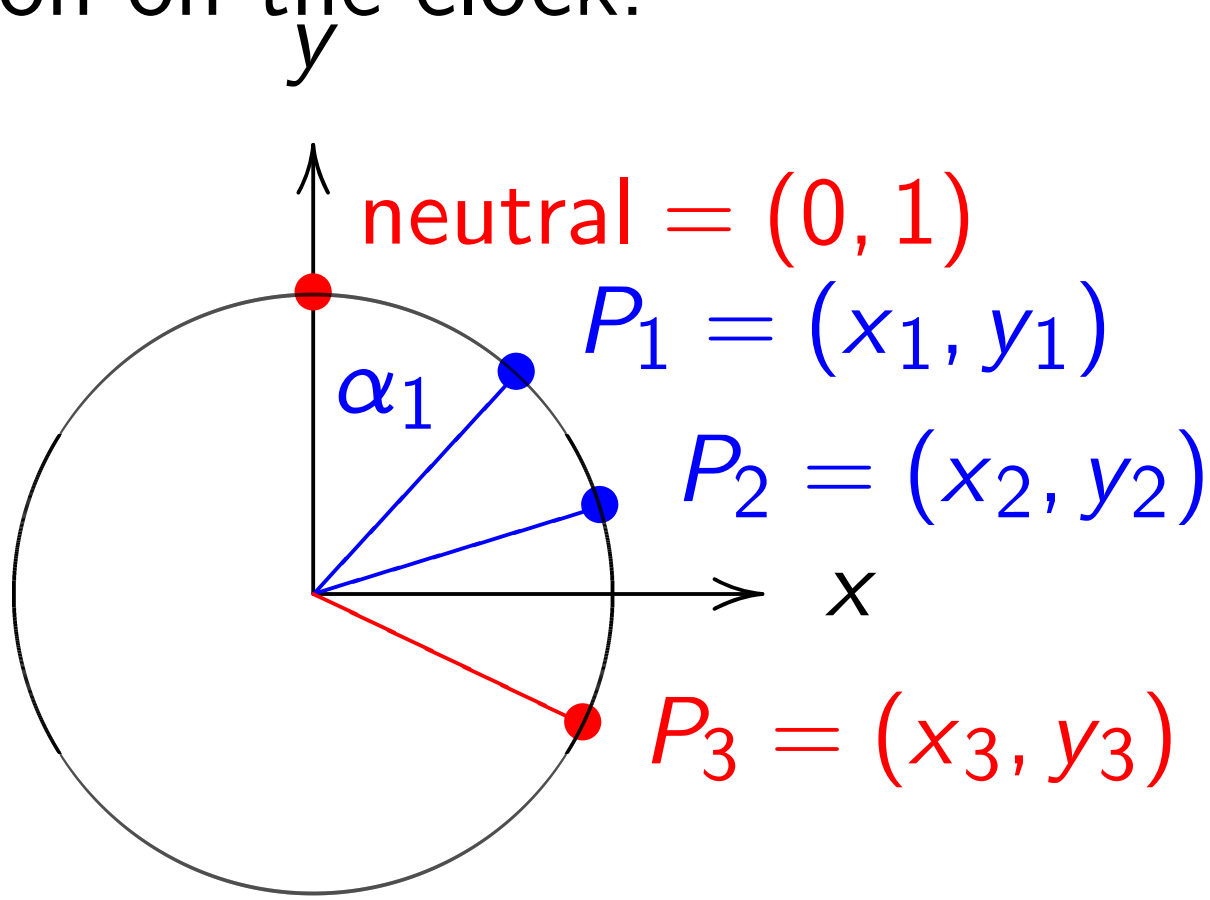

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$$(\sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2,$$
$$\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2).$$

Adding two points corresponds
to adding the angles α_1 and α_2.
Angles modulo 360° are a group,
so points on clock are a group.

Neutral element: angle $\alpha = 0$;
point $(0, 1)$; "12:00".
The point with $\alpha = 180^\circ$
has order 2 and equals 6:00.
3:00 and 9:00 have order 4.
Inverse of point with α
is point with $-\alpha$
since $\alpha + (-\alpha) = 0$.
There are many more points
where angle α is not "nice."

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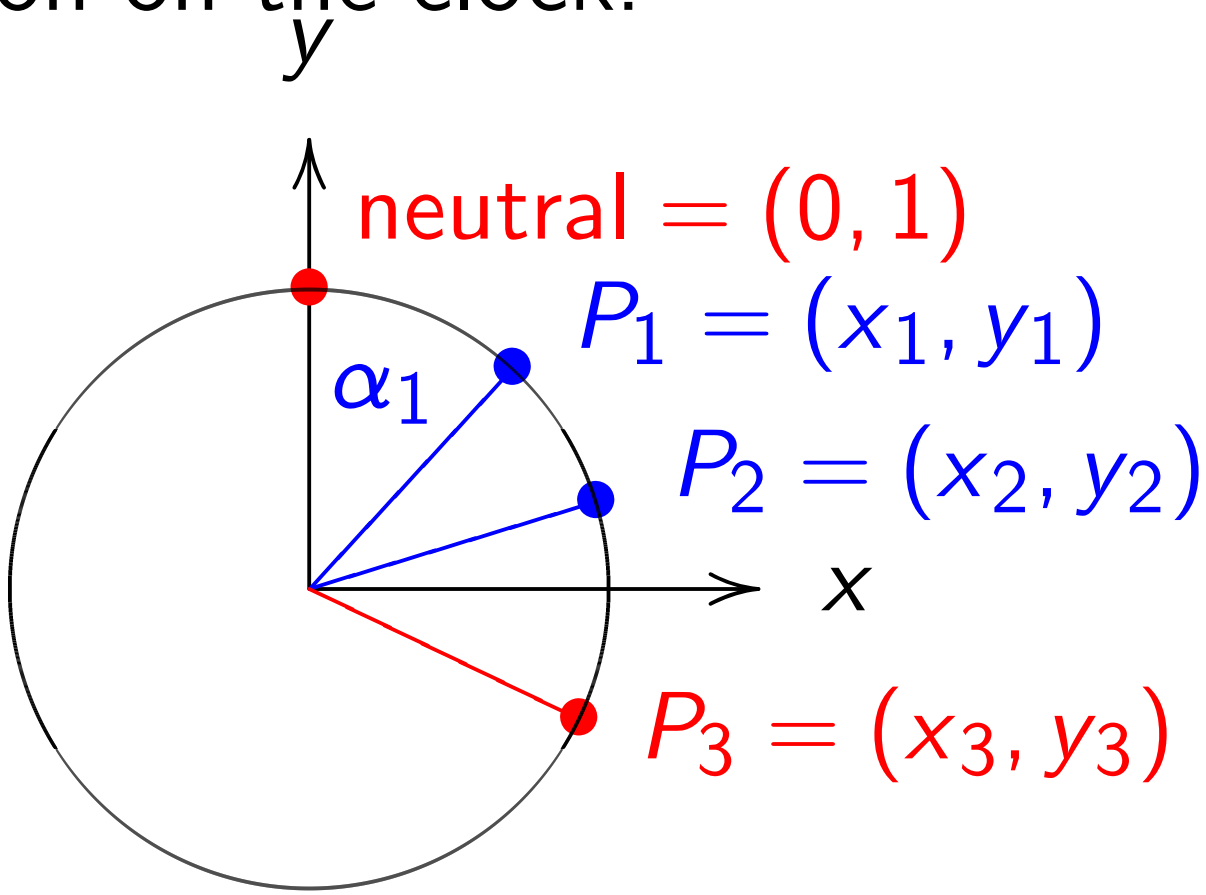

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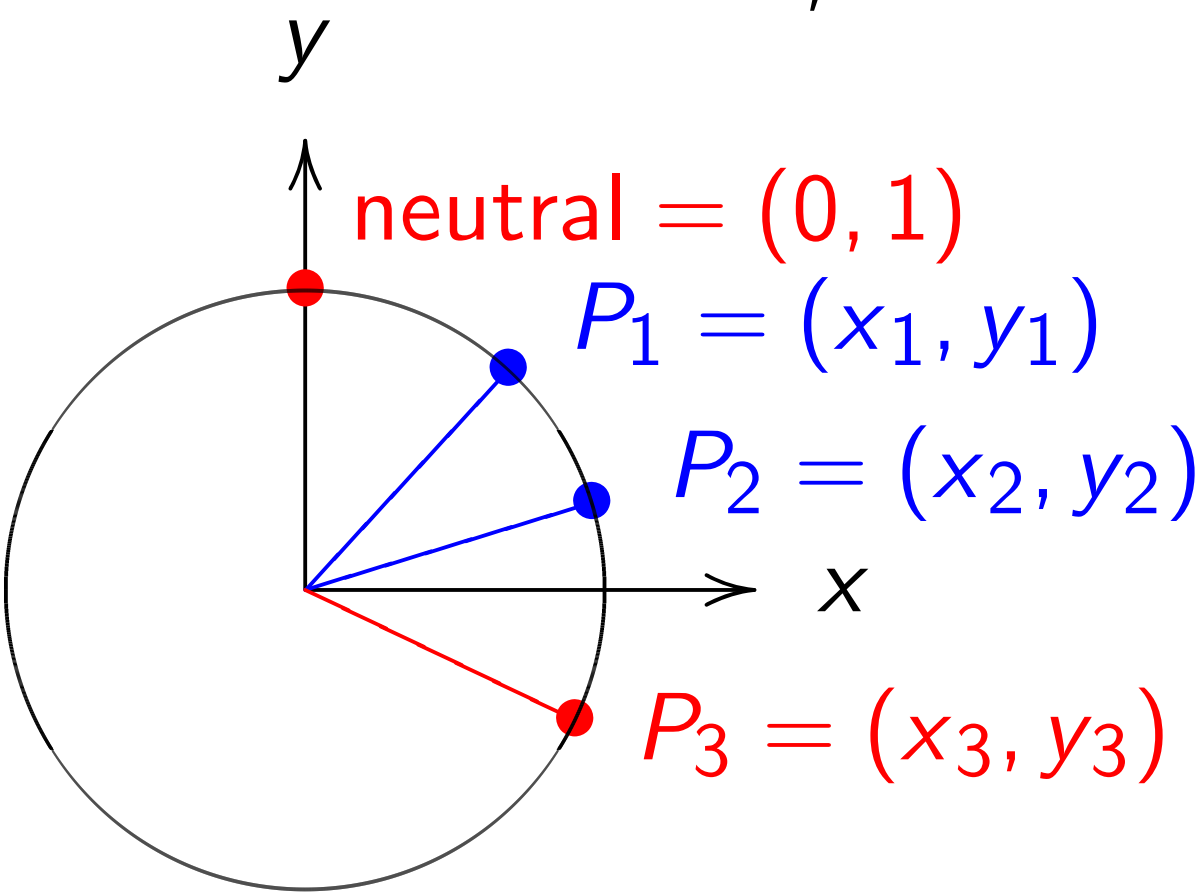
Clock addition without sin, cos:



Use Cartesian coordinates for addition. Addition formula for the clock $x^2 + y^2 = 1$:

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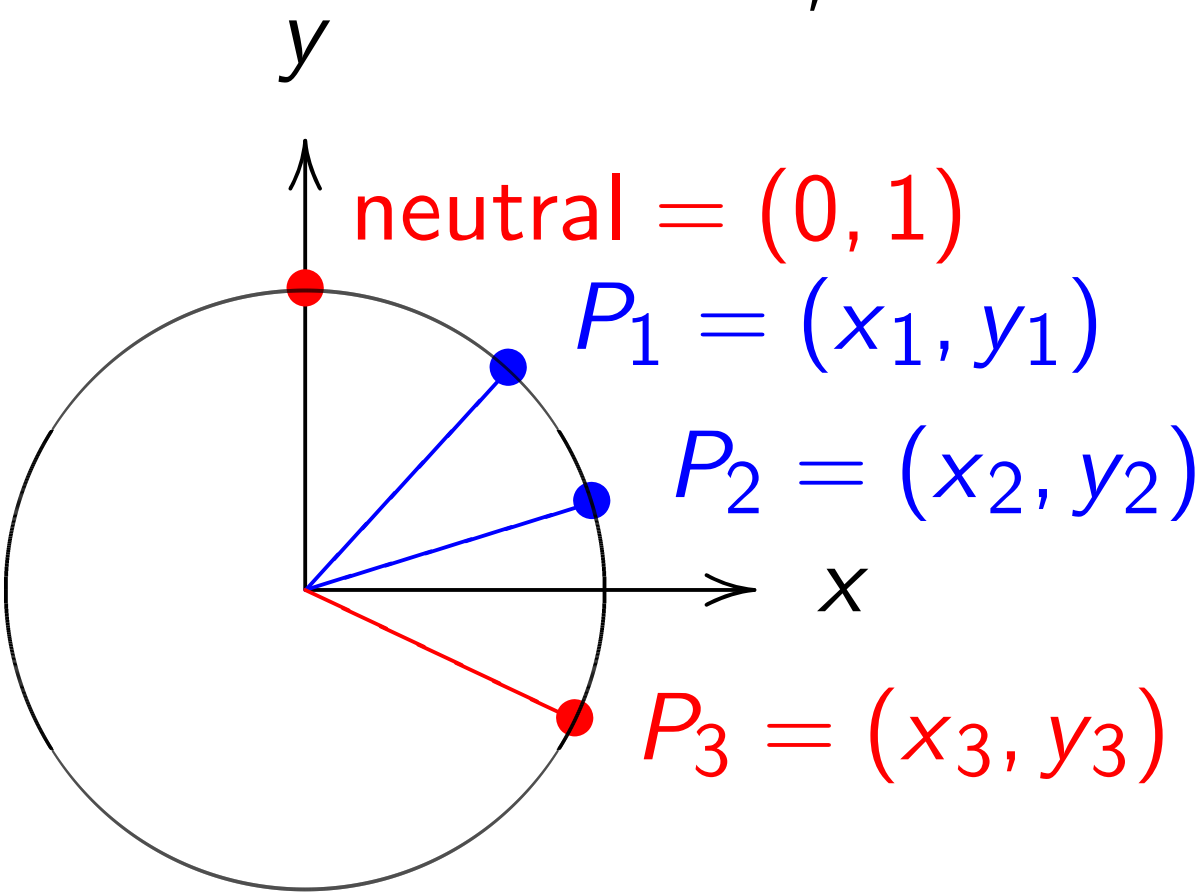

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$= (x_1y_2 + y_1x_2, y_1y_2 - x_1x_2)$.

Note $(x_1, y_1) + (-x_1, y_1) = (0, 1)$.

$kP = \underbrace{P + P + \cdots + P}_{k \text{ copies}}$ for $k \geq 0$.

Examples of clock addition:

$$\text{“2:00”} + \text{“5:00”}$$
$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{“7:00”}.$$

$$\text{“5:00”} + \text{“9:00”}$$
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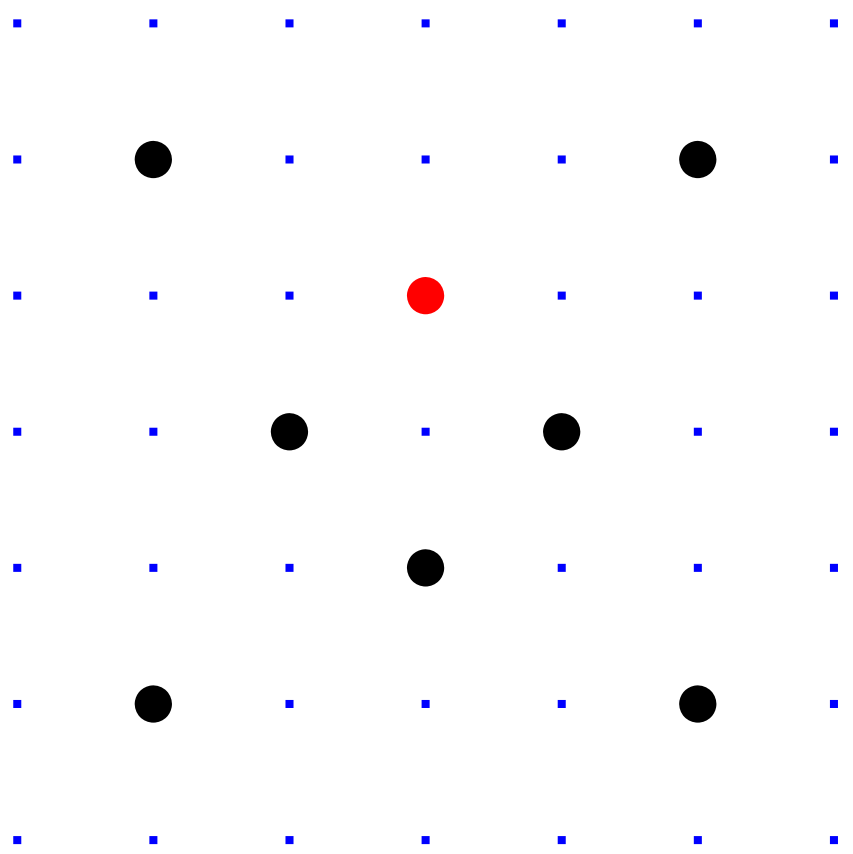
Clocks over finite fields



$\mathrm{Clock}(\mathbf{F}_7) =$

$\{(x, y) \in \mathbf{F}_7 \times \mathbf{F}_7 : x^2 + y^2 = 1\}.$

Here $\mathbf{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

$= \{0, 1, 2, 3, -3, -2, -1\}$

with $+, -, \times$ modulo 7.

E.g. $2 \cdot 5 = 3$ and $3/2 = 5$ in $\mathbf{F}_7$.

Clock cryptography

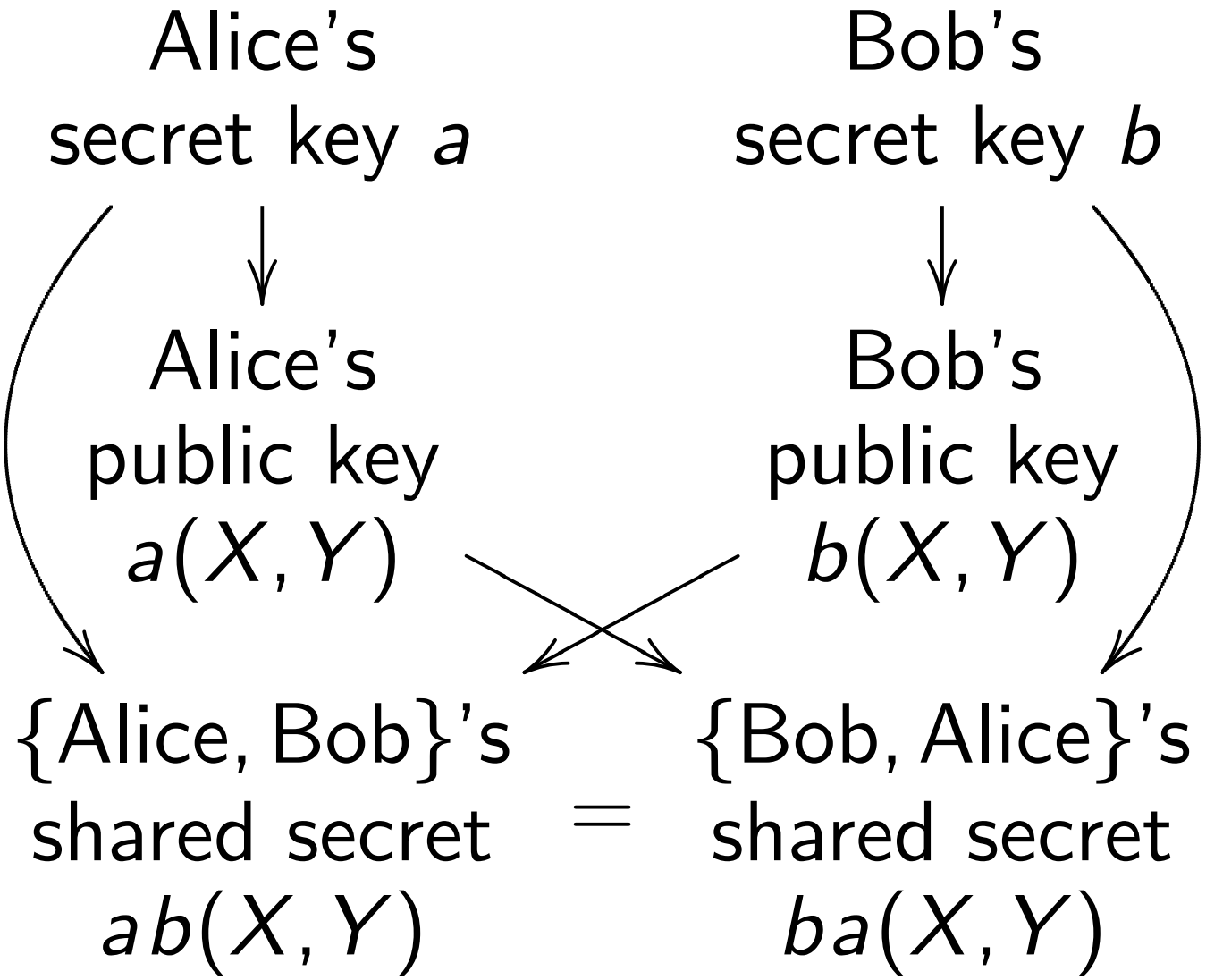
The "Clock Diffie–Hellman protocol":

Standardize large prime p & **base point** $(x, y) \in \mathrm{Clock}(\mathbf{F}_p)$.

Alice chooses big secret a, computes her public key $a(x, y)$.

Bob chooses big secret b, computes his public key $b(x, y)$.

Alice computes $a(b(x, y))$.
Bob computes $b(a(x, y))$.
They use this shared secret to encrypt with AES-GCM etc.


Alice's
secret key a
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Bob's
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{Alice, Bob}'s
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=
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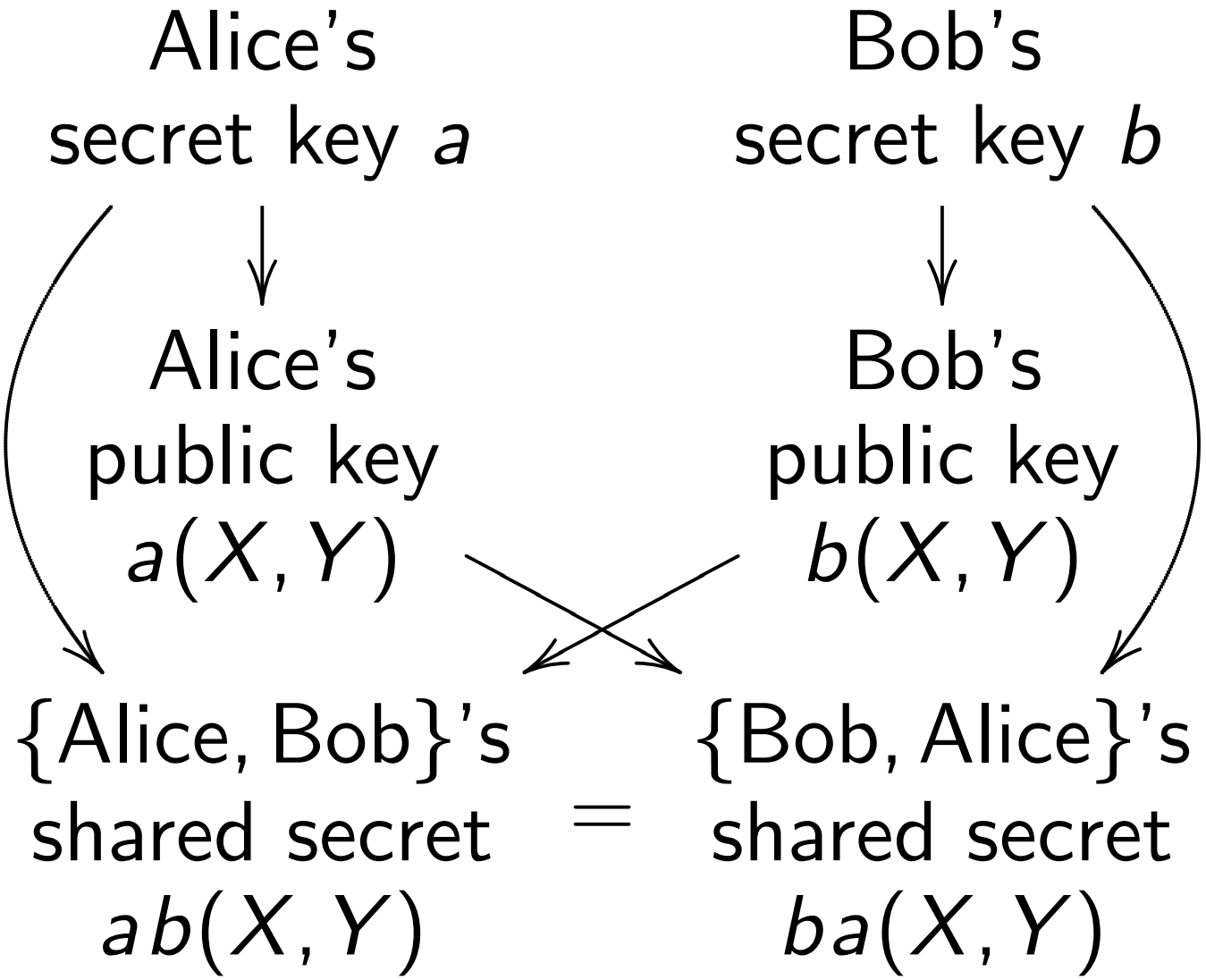


Warning #1:

Many p are unsafe!

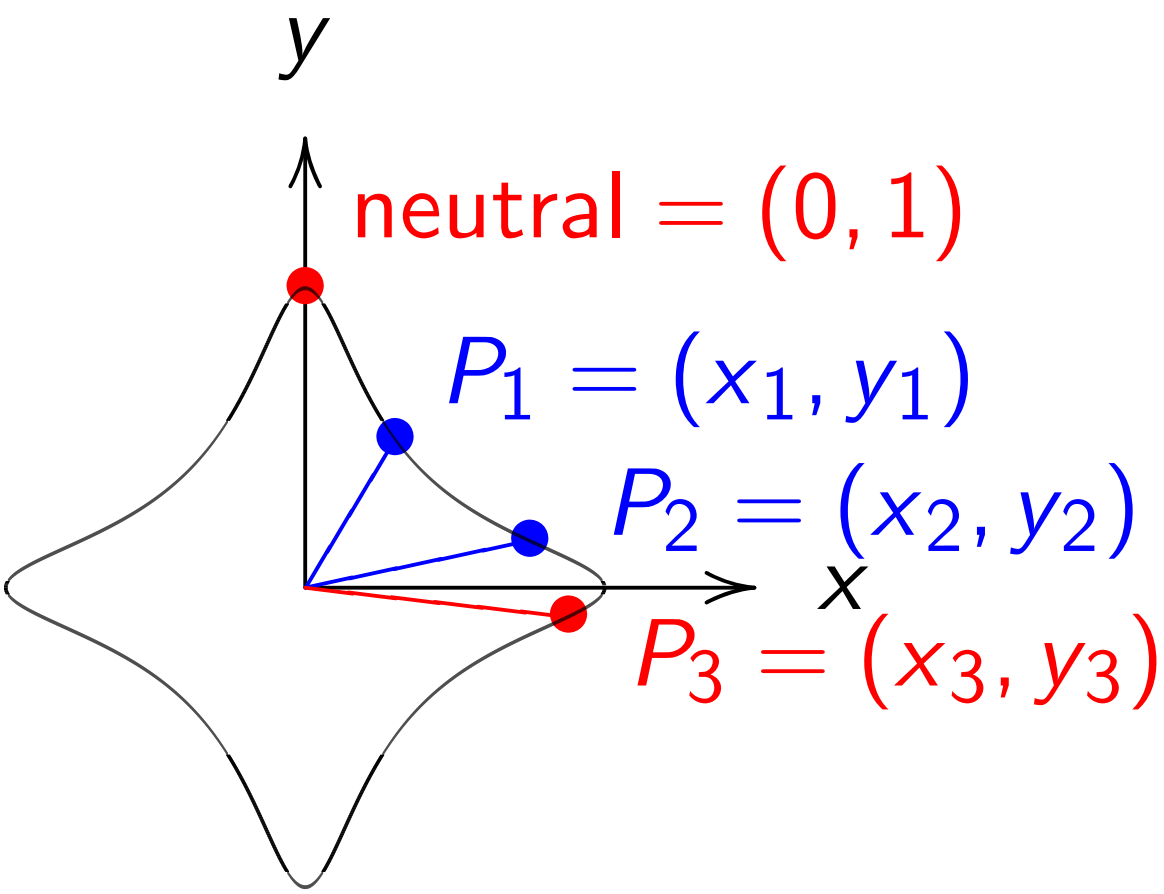
Warning #2:

Clocks aren't elliptic!

To match RSA-3072 security need $p \approx 2^{1536}$.

Addition on an Edwards curve

Change the curve on which Alice and Bob work.

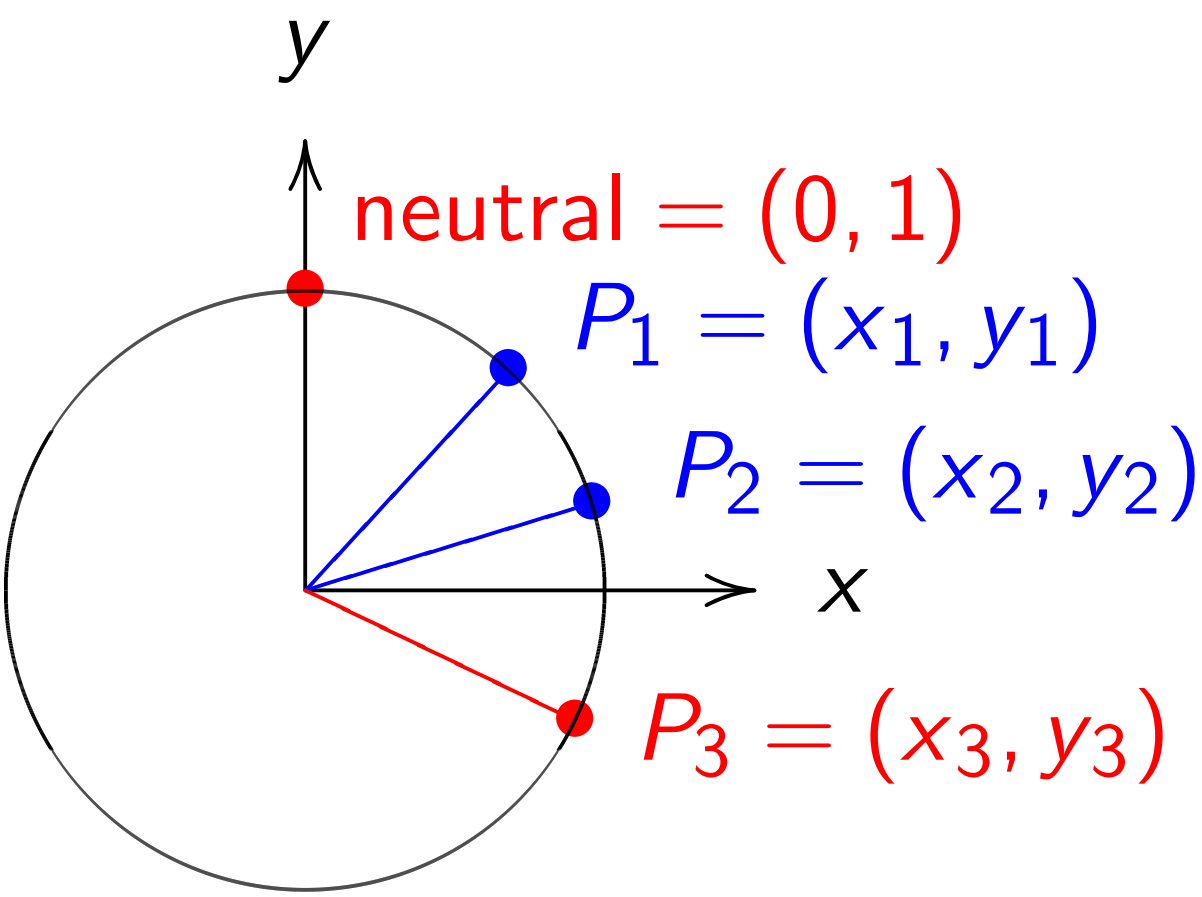


$x^2 + y^2 = 1 - 30x^2y^2$.

Sum of (x_1, y_1) and (x_2, y_2) is

$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2),$
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$.

The clock again, for comparison:



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$y_1y_2 - x_1x_2)$.

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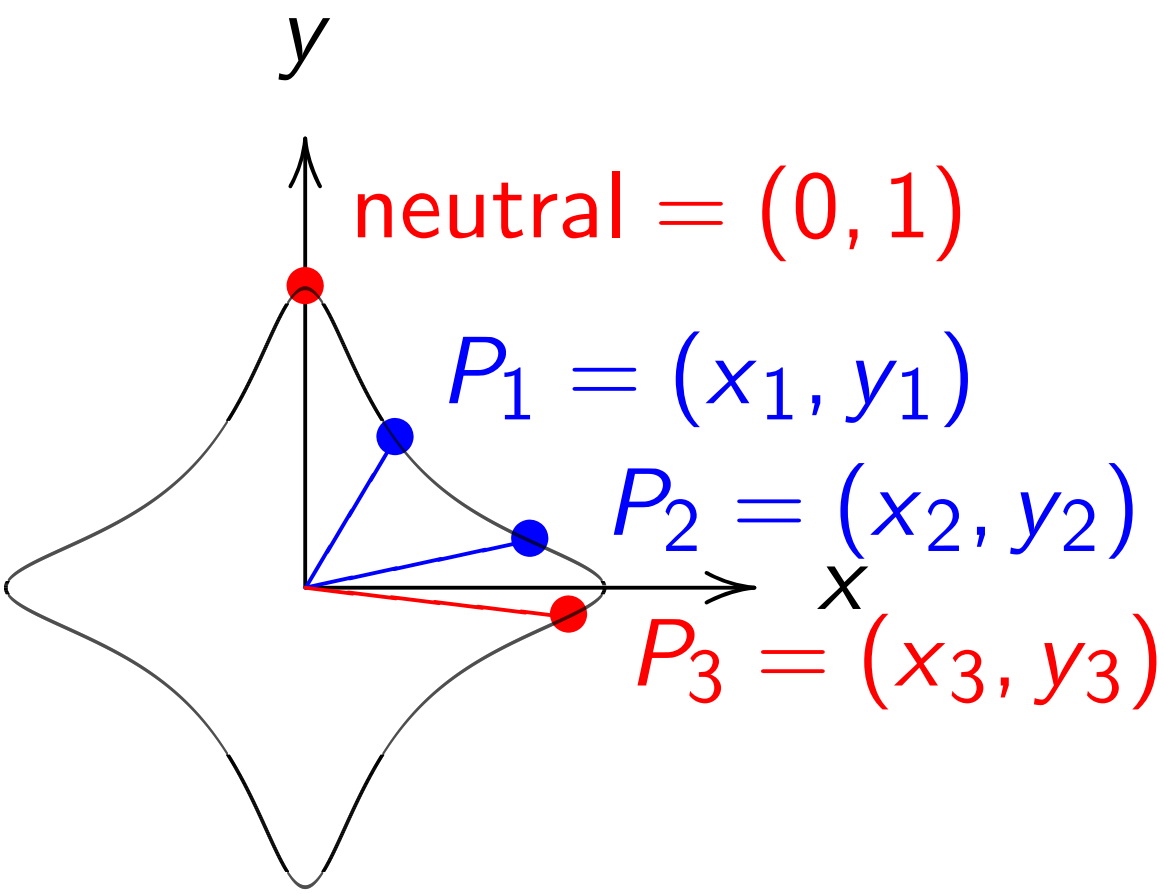

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"Hey, there were divisions
in the Edwards addition law!
What if the denominators are 0?"
Answer: They aren't!
If $x_i = 0$ or $y_i = 0$ then
$1 \pm 30x_1x_2y_1y_2 = 1 \neq 0$.
If $x^2 + y^2 = 1 - 30x^2y^2$
then $30x^2y^2 < 1$
so $\sqrt{30}\ |xy| < 1$.

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If $x_1^2 + y_1^2 = 1 - 30x_1^2y_1^2$
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then $\sqrt{30}\,|x_1y_1| < 1$
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then $\sqrt{30}\,|x_1y_1| < 1$
and $\sqrt{30}\,|x_2y_2| < 1$
so $30\,|x_1y_1x_2y_2| < 1$
so $1 \pm 30x_1x_2y_1y_2 > 0$.

The Edwards addition law
$(x_1, y_1) + (x_2, y_2) =$
$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2),$
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$
is a group law for the curve
$x^2 + y^2 = 1 - 30x^2y^2$.

Some calculation required:

addition result is on curve;

addition law is associative.

Other parts of proof are easy:

addition law is commutative;

$(0, 1)$ is neutral element;

$(x_1, y_1) + (-x_1, y_1) = (0, 1)$.

Edwards curves mod p

Choose an odd prime p.

Choose a *non-square* $d \in \mathbf{F}_p$.

$\{(x, y) \in \mathbf{F}_p \times \mathbf{F}_p :$

$$x^2 + y^2 = 1 + dx^2y^2\}$$

is a "complete Edwards curve".

Roughly $p + 1$ pairs (x, y).

```
def edwardsadd(P1,P2):
  x1,y1 = P1
  x2,y2 = P2
  x3 = (x1*y2+y1*x2)/ \
  (1+d*x1*x2*y1*y2)
  y3 = (y1*y2-x1*x2)/ \
  (1-d*x1*x2*y1*y2)
  return x3,y3
```

Denominators are never 0.

But need different proof;

"$x^2 + y^2 > 0$" doesn't work.

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Addition law is **complete**.

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Answer: Can prove that
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Addition law is **complete**.

This proof relies on
choosing *non-square* d.

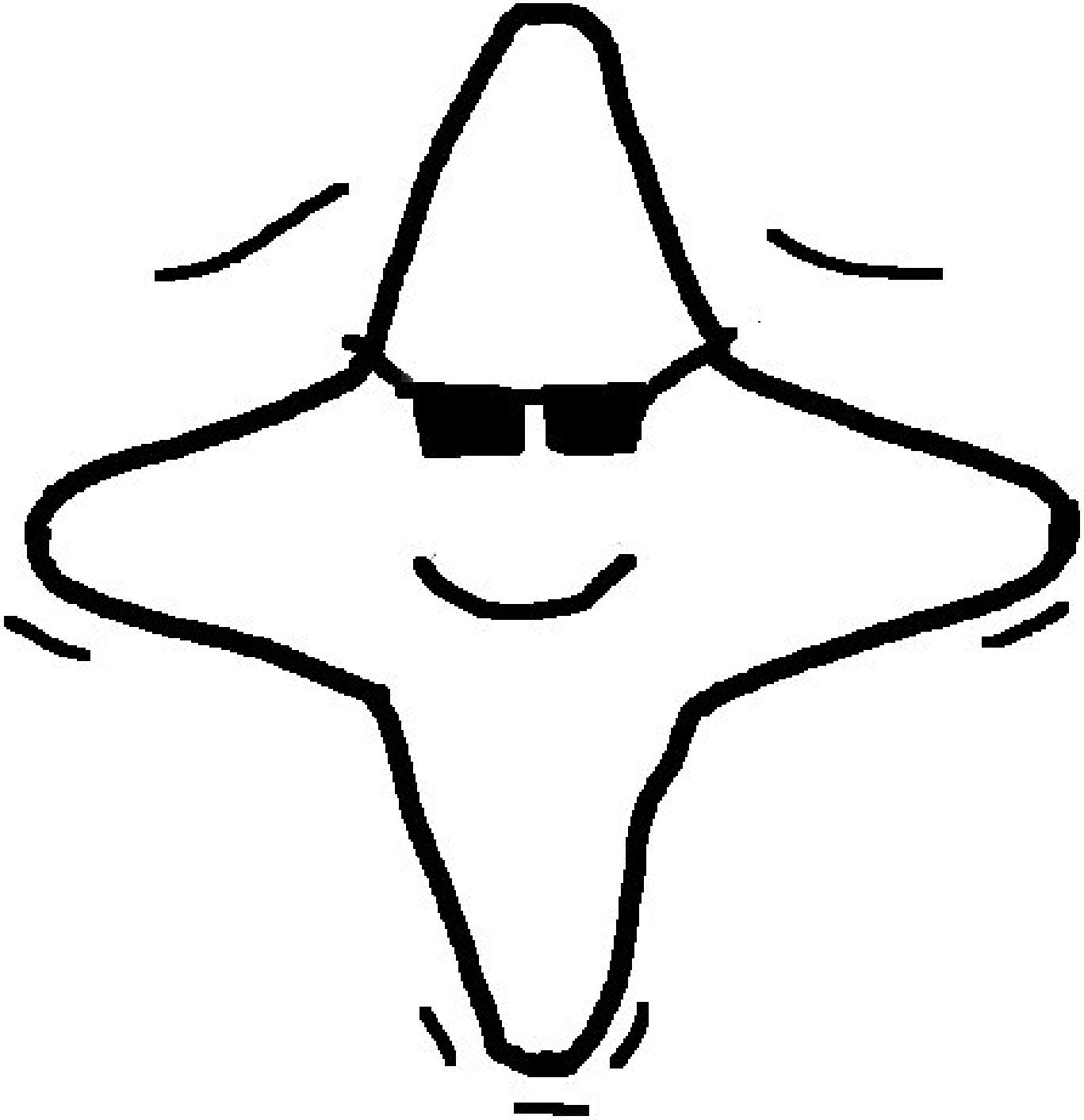
Denominators are never 0.
But need different proof;
"$x^2 + y^2 > 0$" doesn't work.

Answer: Can prove that
the denominators are never 0.
Addition law is **complete**.

This proof relies on
choosing *non-square* d.

If we instead choose square d:
curve is still elliptic, and
addition *seems to work*,
but there are failure cases,
often exploitable by attackers.
Safe code is more complicated.

Edwards curves are cool



ECDSA

Users can sign messages
using Edwards curves.

Take a point P on an Edwards
curve modulo a prime $p > 2$.

ECDSA signer needs to know
the *order of* P.

There are only finitley many other
points; about p in total.
Adding P to itself will eventually
reach $(0, 1)$; let ℓ be the smallest
integer > 0 with $\ell P = (0, 1)$.
This ℓ is the order of P.

The signature scheme has as system parameters a curve E; a base point P; and a hash function h with output length at least $\lfloor \log_2 \ell \rfloor + 1$.
Alice's secret key is an integer a and her public key is $P_A = aP$.

To sign message m,
Alice computes $h(m)$;
picks random k;
computes $R = kP = (x_1, y_1)$;
puts $r \equiv y_1 \bmod \ell$; computes $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.

The signature on m is (r, s).

Anybody can verify signature
given m and (r, s):
Compute $w_1 \equiv s^{-1} h(m) \bmod \ell$
and $w_2 \equiv s^{-1} \cdot r \bmod \ell$.
Check whether the y-coordinate
of $w_1 P + w_2 P_A$ equals r modulo ℓ
and if so, accept signature.

Alice's signatures are valid:

$$w_1 P + w_2 P_A =$$
$$(s^{-1} h(m)) P + (s^{-1} \cdot r) P_A =$$
$$(s^{-1} (h(m) + r a)) P = kP$$

and so the y-coordinate of this
expression equals r,
the y-coordinate of kP.

Attacker's view on signatures

Anybody can produce an $R = kP$.
Alice's private key is only used in
$s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.

Can fake signatures if one can break the DLP, i.e., if one can compute a from P_A.

Sometimes attacks are easier...

If k is known for some $m, (r, s)$ then $a \equiv (sk - h(m))/r \bmod \ell$.

If two signatures $m_1, (r, s_1)$ and $m_2, (r, s_2)$ have the same value for r: assume $k_1 = k_2$; observe $s_1 - s_2 = k_1^{-1}(h(m_1) + ra - (h(m_2) + ra))$; compute $k = (s_1 - s_2)/(h(m_1) - h(m_2))$. Continue as above.

If bits of many k's are known (biased PRNG) can attack $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$ as hidden number problem using lattice basis reduction.

Malicious signer

Alice can set up her public key so that two messages of her choice share the same signature, i.e., she can claim to have signed m_1 or m_2 at will:
$R = (x_1, y_1)$ and $-R = (-x_1, y_1)$ have the same y-coordinate.
Thus, (r, s) fits $R = kP$, $s \equiv k^{-1}(h(m_1) + r\,a) \bmod \ell$ and $-R = (-k)P$, $s \equiv -k^{-1}(h(m_2) + r\,a) \bmod \ell$ if $a \equiv -(h(m_1)+h(m_2))/2r \bmod \ell$.

Malicious signer

Alice can set up her public key so that two messages of her choice share the same signature, i.e., she can claim to have signed m_1 or m_2 at will:
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$a \equiv -(h(m_1)+h(m_2))/2r \bmod \ell$.

(Easy tweak: include bit of x_1.)

More elliptic curves

Edwards curves are elliptic.
Easiest way to understand
elliptic curves is Edwards.

Geometrically, all elliptic curves
are Edwards curves.

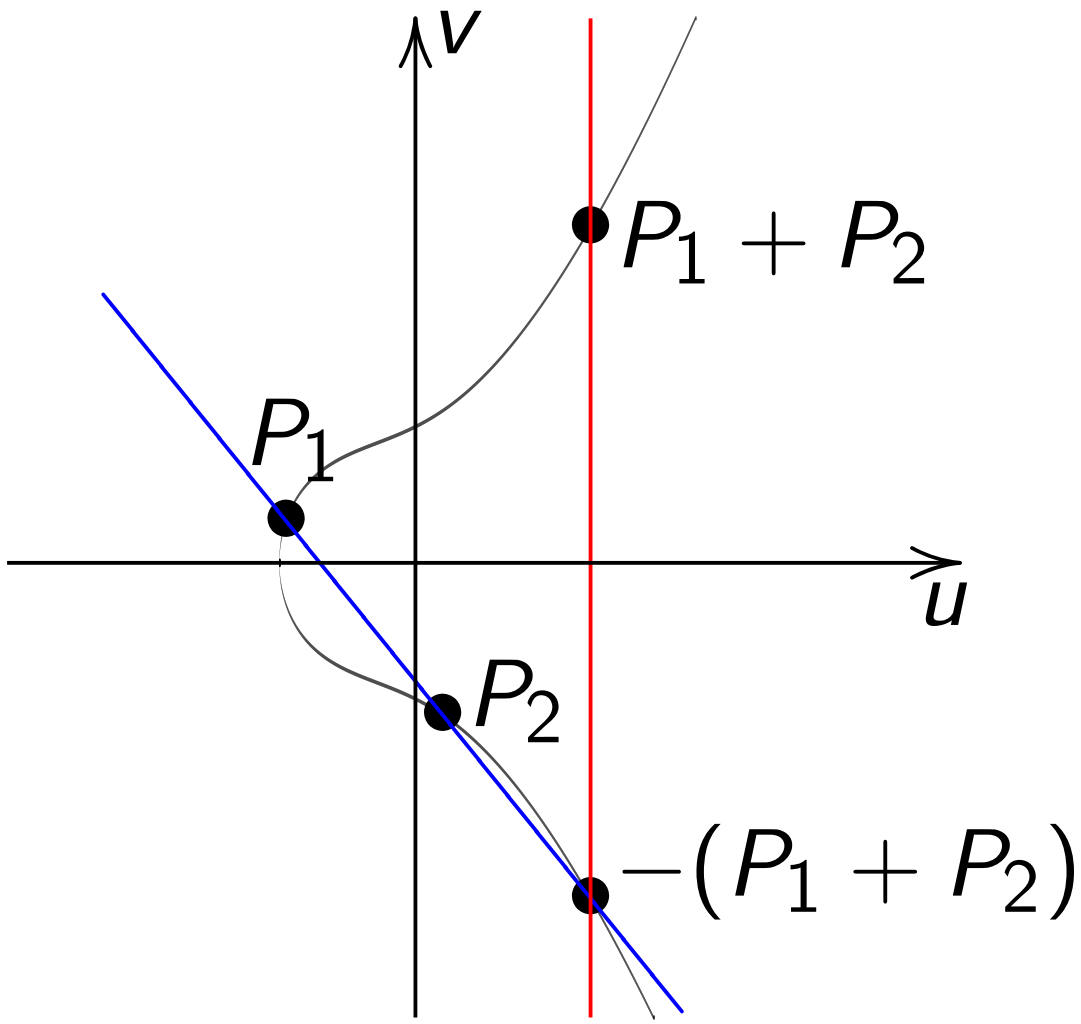
Algebraically,
more elliptic curves exist
(not always point of order 4).

Every odd-char curve can be
expressed as Weierstrass curve
$v^2 = u^3 + a_2u^2 + a_4u + a_6$.

Warning: "Weierstrass" has
different meaning in char 2.

Addition on Weierstrass curve

$$v^2 = u^3 + u^2 + u + 1$$

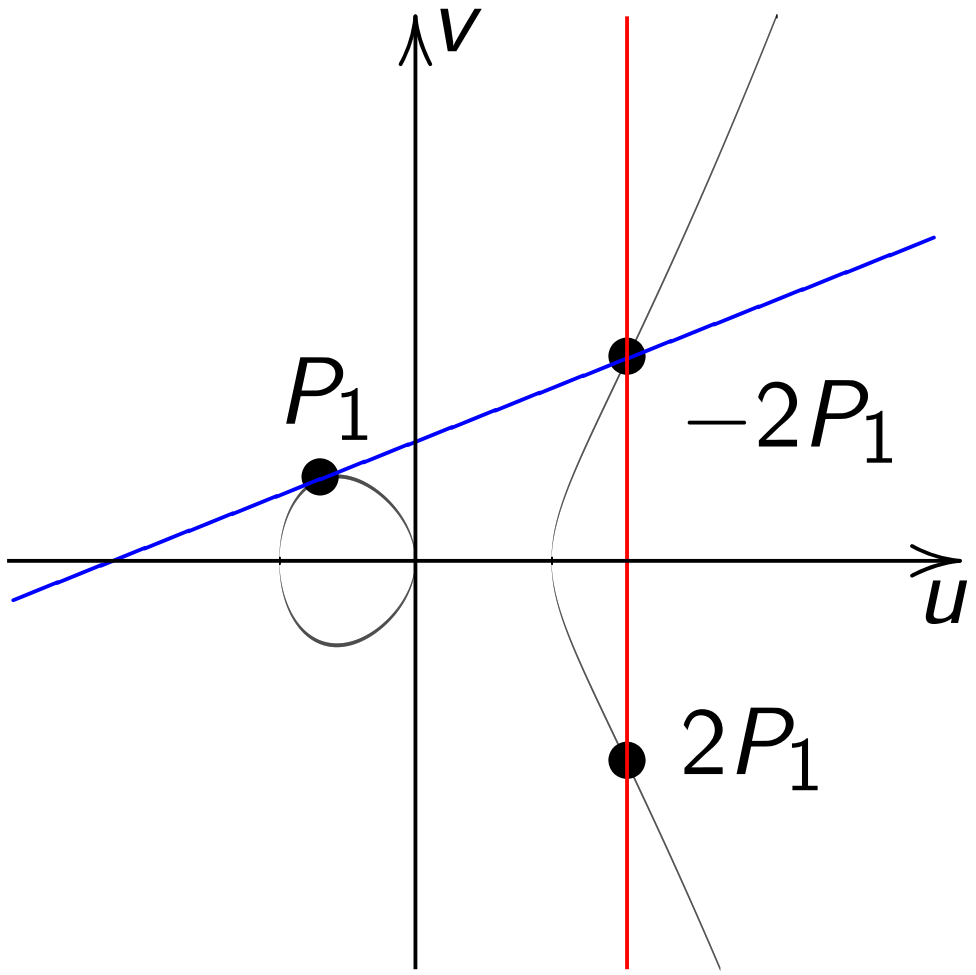


Slope $\lambda = (v_2 - v_1)/(u_2 - u_1)$.

Note that $u_1 \neq u_2$.

Doubling on Weierstrass curve

$v^2 = u^3 - u$



Slope $\lambda = (3u_1^2 - 1)/(2v_1)$.

In most cases
$(u_1, v_1) + (u_2, v_2) =$
(u_3, v_3) where $(u_3, v_3) =$
$(\lambda^2 - u_1 - u_2, \lambda(u_1 - u_3) - v_1)$.

$u_1 \neq u_2$, "addition" (alert!):
$\lambda = (v_2 - v_1)/(u_2 - u_1)$.
Total cost $1\mathbf{I} + 2\mathbf{M} + 1\mathbf{S}$.

$(u_1, v_1) = (u_2, v_2)$ and $v_1 \neq 0$,
"doubling" (alert!):
$\lambda = (3u_1^2 + 2a_2u_1 + a_4)/(2v_1)$.
Total cost $1\mathbf{I} + 2\mathbf{M} + 2\mathbf{S}$.

Also handle some exceptions:
$(u_1, v_1) = (u_2, -v_2)$; ∞ as input.
Messy to implement and test.

Birational equivalence

Starting from point (x, y)
on $x^2 + y^2 = 1 + dx^2y^2$:

Define $A = 2(1 + d)/(1 - d)$,
$B = 4/(1 - d)$;
$u = (1 + y)/(B(1 - y))$,
$v = u/x = (1 + y)/(Bx(1 - y))$.
(Skip a few exceptional points.)

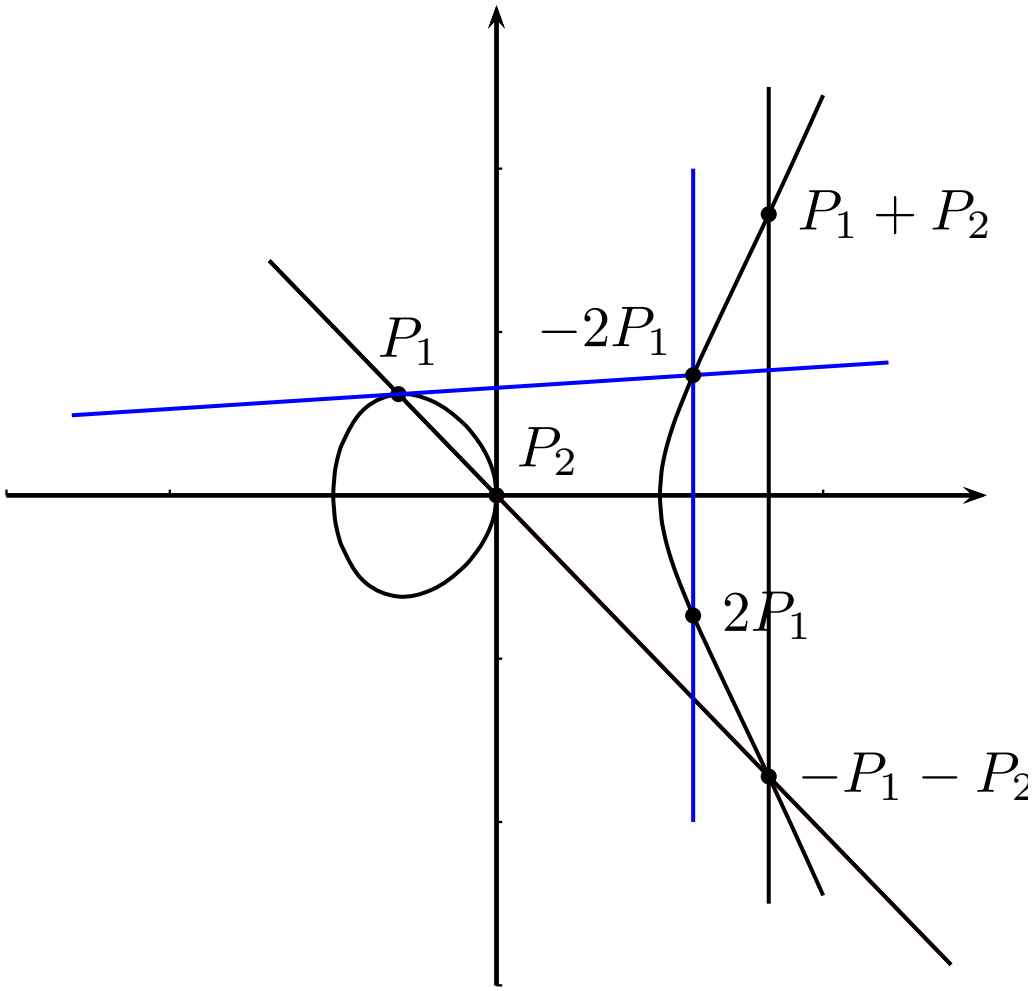
Then (u, v) is a point on
a Weierstrass curve:
$v^2 = u^3 + (A/B)u^2 + (1/B^2)u$.

Easily invert this map:
$x = u/v$, $y = (Bu - 1)/(Bu + 1)$.

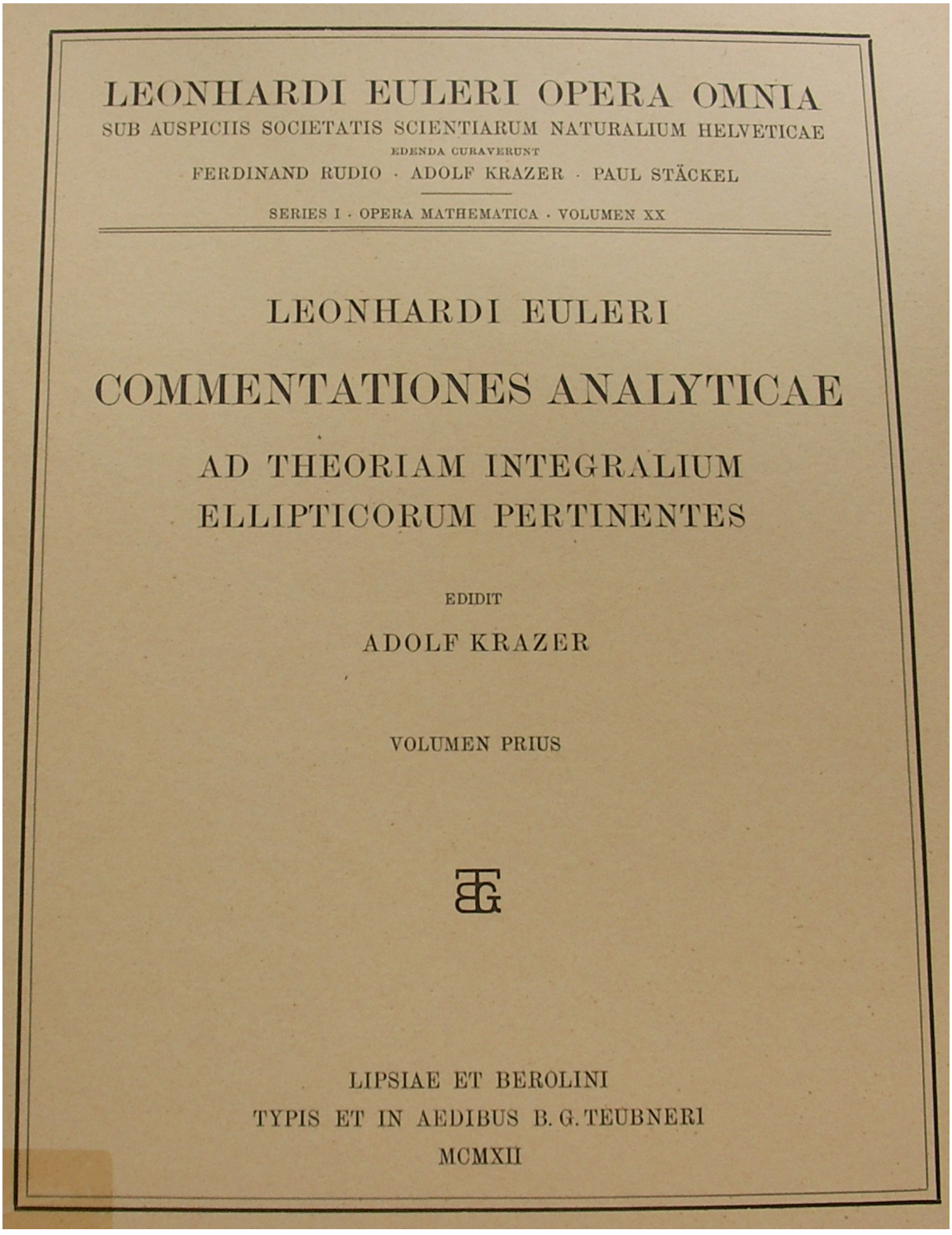
Attacker can transform Edwards curve to Weierstrass curve and vice versa; $n(x, y) \mapsto n(u, v)$.
$\Rightarrow$ Same discrete-log security!
Can choose curve representation so that implementation of attack is faster/easier.

System designer can choose curve representation so that protocol runs fastest; no need to worry about security degradation.

Elliptic-curve groups



History



LEONHARDI EULERI OPERA OMNIA

SUB AUSPICIIS SOCIETATIS SCIENTIARUM NATURALIUM HELVETICAE

EDENDA CURAVERUNT

FERDINAND RUDIO · ADOLF KRAZER · PAUL STÄCKEL

SERIES I · OPERA MATHEMATICA · VOLUMEN XX

LEONHARDI EULERI

COMMENTATIONES ANALYTICAE

AD THEORIAM INTEGRALIUM ELLIPTICORUM PERTINENTES

EDIDIT

ADOLF KRAZER

VOLUMEN PRIUS

LIPSIAE ET BEROLINI

TYPIS ET IN AEDIBUS B. G. TEUBNERI

MCMXII

Euler

Observationes de Comparatione Arcuum Curvarum Irrectificabilium

I. DE ELLIPSI

1. Sit quadrans ellipticus ABC (Fig. 1), cuius centrum in C, eiusque semiaxes ponantur $CA=1$ et $CB=c$; sumta ergo abscissa quacunque $CP=x$ erit applicata ei respondens $PM=y=c\sqrt{(1-xx)}$; cuius differentiale cum sit $dy=-\frac{cxdx}{\sqrt{(1-xx)}}$, erit abscissae $CP=x$ arcus ellipticus respondens

$$BM=\int\frac{dx\sqrt{(1-(1-cc)xx)}}{\sqrt{(1-xx)}}.$$

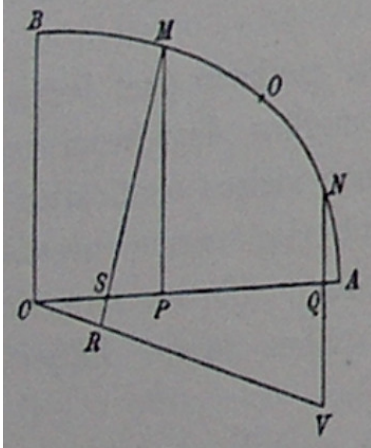


Fig. 1.

Ponatur brevitatis gratia $1-cc=n$, ut sit arcus

$$BM=\int dx\sqrt{\frac{1-nxx}{1-xx}},$$

$1/y = (1 - nx^2)/(1 - x^2)$

matches

$x^2 + y^2 = 1 + nx^2y^2.$

Gauss

[2.]

$c + ssc\,c$ sive $2 = (1+ss)(1+c\,c) = (\frac{1}{ss}$ —

$$s = \sqrt{\frac{1-ee}{1+ee}}, \qquad c = \sqrt{\frac{1-ss}{1+ss}}$$

$$\sin \operatorname{lemn}(a \pm b) = \frac{sc' \pm s'c}{1 \mp sc\,s'c'}$$

$$\cos \operatorname{lemn}(a \pm b) = \frac{cc' \mp ss'}{1 \pm ss'cc'}$$

$-a) = -\sin \operatorname{lemn} a, \qquad \cos \operatorname{lemn}(-a) = \cos$

$\operatorname{lemn} k\varpi = 0 \qquad \sin \operatorname{lemn}(k + \frac{1}{2})\varpi = \pm 1$

$\operatorname{lemn} k\varpi = \pm 1 \qquad \cos \operatorname{lemn}(k + \frac{1}{2})\varpi = 0$

General addition formulas for

$1 = s^2 + c^2 + s^2c^2$

Harold M. Edwards

Bulletin of the AMS,

44, 393–422, 2007

Every elliptic curve

can be written as

$x^2 + y^2 = a^2(1 + x^2y^2), a^5 \neq a$

over some extension field.

Security requirements

We want elliptic curve $E/\mathbf{F}_p$ with $\#E(\mathbf{F}_p)$ almost prime,
e.g., $\#E(\mathbf{F}_p) = 4 \cdot \ell$.

$p \approx \ell$ of ≈ 256 bits.

Other conditions:
E should be ordinary, i.e.,
$\#E(\mathbf{F}_p) \neq p + 1$.
E should not be anomalous,
i.e., $\#E(\mathbf{F}_p) \neq p$.
For more properties,
and considerations about
secure implementations see
https://safecurves.cr.yp.to/