Code-Based Cryptography

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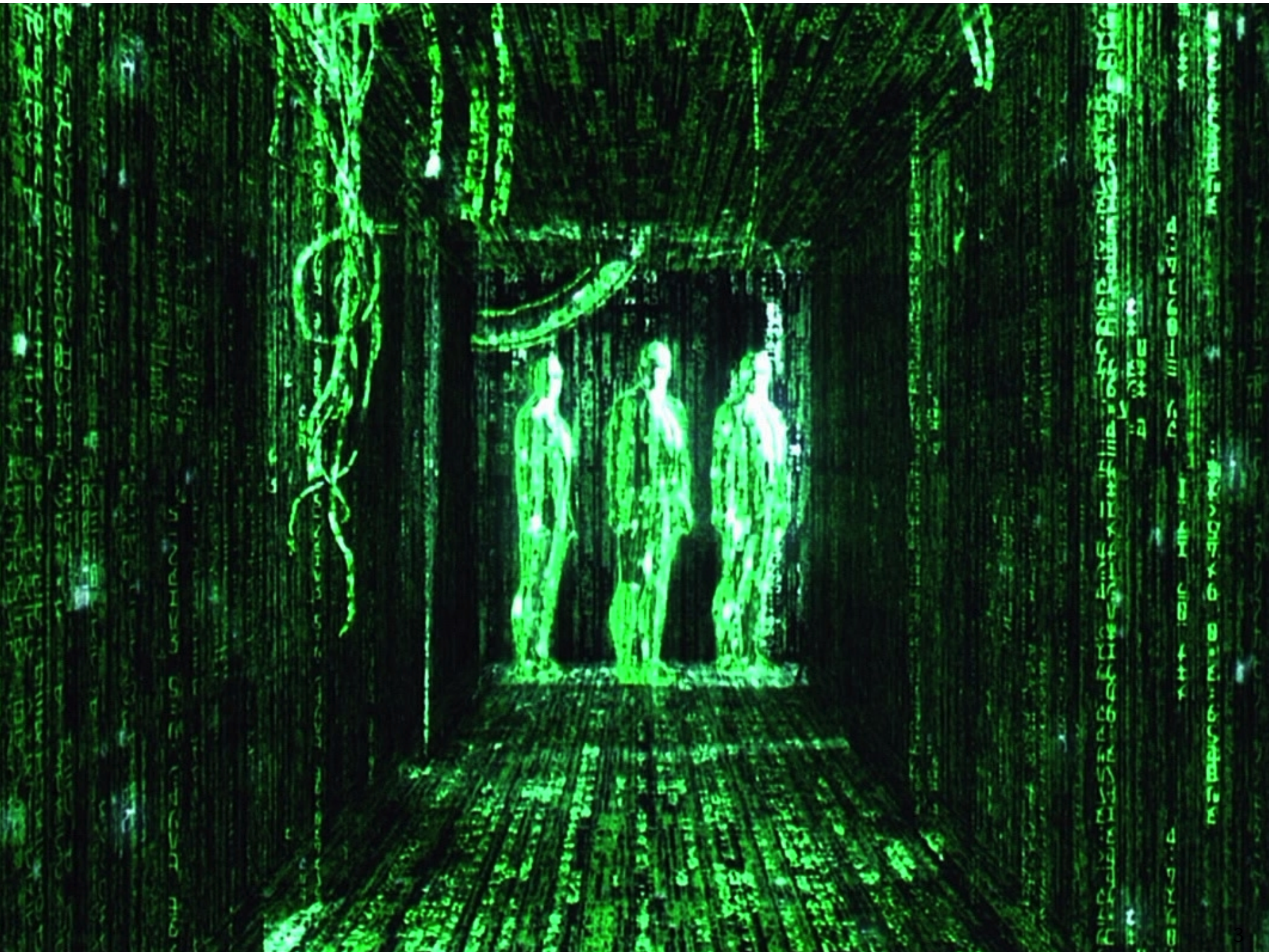
Error correction

- Digital media is exposed to memory corruption.
- Many systems check whether data was corrupted in transit:
  - ISBN numbers have check digit to detect corruption.
  - ECC RAM detects up to two errors and can correct one error. 64 bits are stored as 72 bits: extra 8 bits for checks and recovery.
- In general, k bits of data get stored in n bits, adding some redundancy.
- If no error occurred, these n bits satisfy $n - k$ parity check equations; else can correct errors from the error pattern.
- Good codes can correct many errors without blowing up storage too much;
  offer guarantee to correct t errors (often can correct or at least detect more).
- To represent these check equations we need a matrix.



Hamming code

Parity check matrix ($n = 7, k = 4$):

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

An error-free string of 7 bits $\mathbf{b} = (b_0, b_1, b_2, b_3, b_4, b_5, b_6)$ satisfies these three equations:

$$\begin{array}{rrrrrrrrr} b_0 & +b_1 & & +b_3 & +b_4 & & & = & 0 \\ b_0 & & +b_2 & +b_3 & & +b_5 & & = & 0 \\ & b_1 & +b_2 & +b_3 & & & +b_6 & = & 0 \end{array}$$

If one error occurred at least one of these equations will not hold.
Failure pattern uniquely identifies the error location,
e.g., $1, 0, 1$ means

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e.g., $1, 0, 1$ means b_1 flipped.
In math notation, the failure pattern is $H \cdot \mathbf{b}$.

Coding theory

- Names: code word $\mathbf{c}$, error vector $\mathbf{e}$, received word $\mathbf{b} = \mathbf{c} + \mathbf{e}$.
- Very common to transform the matrix so that the right part has just 1 on the diagonal (no need to store that).

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

- Many special constructions discovered in 65 years of coding theory:
  - Large matrix H.
  - Fast decoding algorithm to find $\mathbf{e}$ given $\mathbf{s} = H \cdot (\mathbf{c} + \mathbf{e})$, whenever $\mathbf{e}$ does not have too many bits set.
- Given large H, usually very hard to find fast decoding algorithm.
- Use this difference in complexities for encryption.

Code-based encryption

- 1971 Goppa: Fast decoders for many matrices H.
- 1978 McEliece: Use Goppa codes for public-key crypto.
  - Original parameters designed for 2^{64} security.
  - 2008 Bernstein–Lange–Peters: broken in $\approx 2^{60}$ cycles.
  - Easily scale up for higher security.
- 1986 Niederreiter: Simplified and smaller version of McEliece.
- 1962 Prange: simple attack idea guiding sizes in 1978 McEliece.
  The McEliece system (with later key-size optimizations)
  uses $(c_0 + o(1))\lambda^2(\lg \lambda)^2$-bit keys as $\lambda \to \infty$
  to achieve 2^λ security against Prange's attack.
  Here $c_0 \approx 0.7418860694$.

Security analysis

Some papers studying algorithms for attackers:
1962 Prange; 1981 Clark–Cain, crediting Omura; 1988 Lee–Brickell; 1988 Leon; 1989 Krouk; 1989 Stern; 1989 Dumer; 1990 Coffey–Goodman; 1990 van Tilburg; 1991 Dumer; 1991 Coffey–Goodman–Farrell; 1993 Chabanne–Courteau; 1993 Chabaud; 1994 van Tilburg; 1994 Canteaut–Chabanne; 1998 Canteaut–Chabaud; 1998 Canteaut–Sendrier; 2008 Bernstein–Lange–Peters; 2009 Bernstein–Lange–Peters–van Tilborg; 2009 Bernstein (**post-quantum**); 2009 Finiasz–Sendrier; 2010 Bernstein–Lange–Peters; 2011 May–Meurer–Thomae; 2012 Becker–Joux–May–Meurer; 2013 Hamdaoui–Sendrier; 2015 May–Ozerov; 2016 Canto Torres–Sendrier; 2017 Kachigar–Tillich (**post-quantum**); 2017 Both–May; 2018 Both–May; 2018 Kirshanova (**post-quantum**).

Consequence of security analysis

- The McEliece system (with later key-size optimizations) uses $(c_0 + o(1))\lambda^2(\lg \lambda)^2$-bit keys as $\lambda \to \infty$ to achieve 2^λ security against all these attacks.

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- 256 KB public key for 2^{146} pre-quantum security.
- 512 KB public key for 2^{187} pre-quantum security.
- 1024 KB public key for 2^{263} pre-quantum security.

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- 256 KB public key for 2^{146} pre-quantum security.
- 512 KB public key for 2^{187} pre-quantum security.
- 1024 KB public key for 2^{263} pre-quantum security.
- Post-quantum (Grover): below 2^{263}, above 2^{131}.

Linear codes

A binary linear code C of length n and dimension k is a k-dimensional subspace of $\mathbb{F}_2^n$.
C is usually specified as

- the row space of a generating matrix $G \in \mathbb{F}_2^{k \times n}$

$$C = \{\mathbf{m}G | \mathbf{m} \in \mathbb{F}_2^k\}$$

- the kernel space of a parity-check matrix $H \in \mathbb{F}_2^{(n-k) \times n}$

$$C = \{\mathbf{c} | H\mathbf{c}^\mathsf{T} = 0, \ \ \mathbf{c} \in \mathbb{F}_2^n\}$$

Leaving out the $^\mathsf{T}$ from now on.

Example

$$G = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$\mathbf{c} = (111)G = (10011)$ is a codeword.

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Same with parity-check matrix:

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Same with parity-check matrix:

$$H(\mathbf{c}_1 + \mathbf{c}_2) = H\mathbf{c}_1 + H\mathbf{c}_2 = 0 + 0 = 0.$$

Hamming weight and distance

- The Hamming weight of a word is the number of nonzero coordinates.

$$\mathrm{wt}(1,0,0,1,1) = 3$$

- The Hamming distance between two words in $\mathbb{F}_2^n$ is the number of coordinates in which they differ.

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$$d((1,1,0,1,1),(1,0,0,1,1)) = 1$$

The Hamming distance between $\mathbf{x}$ and $\mathbf{y}$ equals the Hamming weight of $\mathbf{x} + \mathbf{y}$:

$$d((1,1,0,1,1),(1,0,0,1,1)) = \mathrm{wt}(0,1,0,0,0).$$

Minimum distance

- The minimum distance of a linear code C is the smallest Hamming weight of a nonzero codeword in C.

$$d = \min_{0 \neq \mathbf{c} \in C} \{\mathrm{wt}(\mathbf{c})\} = \min_{\mathbf{b} \neq \mathbf{c} \in C} \{d(\mathbf{b}, \mathbf{c})\}$$

- In code with minimum distance $d = 2t + 1$, any vector $\mathbf{x} = \mathbf{c} + \mathbf{e}$ with $\mathrm{wt}(\mathbf{e}) \leq t$ is uniquely decodable to $\mathbf{c}$; i. e. there is no closer code word.

Decoding problem

Decoding problem: find the closest codeword $\mathbf{c} \in C$ to a given $\mathbf{x} \in \mathbb{F}_2^n$, assuming that there is a unique closest codeword. Let $\mathbf{x} = \mathbf{c} + \mathbf{e}$. Note that finding $\mathbf{e}$ is an equivalent problem.

- If $\mathbf{c}$ is t errors away from $\mathbf{x}$, i.e., the Hamming weight of $\mathbf{e}$ is t, this is called a t-error correcting problem.
- There are lots of code families with fast decoding algorithms, e.g., Reed–Solomon codes, Goppa codes/alternant codes, etc.
- However, the general decoding problem is hard: Information-set decoding (see later) takes exponential time.

The McEliece cryptosystem I

- Let C be a length-n binary Goppa code Γ of dimension k with minimum distance $2t+1$ where $t \approx (n-k)/\log_2(n)$; original parameters (1978) $n = 1024$, $k = 524$, $t = 50$.
- The McEliece secret key consists of a generator matrix G for Γ, an efficient t-error correcting decoding algorithm for Γ; an $n \times n$ permutation matrix P and a nonsingular $k \times k$ matrix S.
- n, k, t are public; but Γ, P, S are randomly generated secrets.
- The McEliece public key is the $k \times n$ matrix $G' = SGP$.

The McEliece cryptosystem II

- Encrypt: Compute $\mathbf{m}G'$ and add a random error vector $\mathbf{e}$ of weight t and length n. Send $\mathbf{y} = \mathbf{m}G' + \mathbf{e}$.
- Decrypt: Compute $\mathbf{y}P^{-1} = \mathbf{m}G'P^{-1} + \mathbf{e}P^{-1} = (\mathbf{m}S)G + \mathbf{e}P^{-1}$. This works because $\mathbf{e}P^{-1}$ has the same weight as $\mathbf{e}$

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  This works because $\mathbf{e}P^{-1}$ has the same weight as $\mathbf{e}$
  because P is a permutation matrix.
  Use fast decoding to find $\mathbf{m}S$ and $\mathbf{m}$.
- Attacker is faced with decoding $\mathbf{y}$ to nearest codeword $\mathbf{m}G'$ in the code generated by G'.
  This is general decoding if G' does not expose any structure.

Systematic form

- A systematic generator matrix is a generator matrix of the form $(I_k|Q)$ where I_k is the $k \times k$ identity matrix and Q is a $k \times (n-k)$ matrix (redundant part).
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  Then

$$H(\mathbf{m}G)^\mathsf{T} = HG^\mathsf{T}\mathbf{m}^\mathsf{T} = (Q^\mathsf{T}|I_{n-k})(I_k|Q)^\mathsf{T}\mathbf{m}^\mathsf{T} = 0.$$

Different views on decoding

- The syndrome of $\mathbf{x} \in \mathbb{F}_2^n$ is $\mathbf{s} = H\mathbf{x}$.
  Note $H\mathbf{x} = H(\mathbf{c} + \mathbf{e}) = H\mathbf{c} + H\mathbf{e} = H\mathbf{e}$ depends only on $\mathbf{e}$.
- The syndrome decoding problem is to compute $\mathbf{e} \in \mathbb{F}_2^n$ given $\mathbf{s} \in \mathbb{F}_2^{n-k}$ so that $H\mathbf{e} = \mathbf{s}$ and $\mathbf{e}$ has minimal weight.
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  To decode $\mathbf{x}$ with syndrome decoder, compute $\mathbf{e}$ from $H\mathbf{x}$, then $\mathbf{c} = \mathbf{x} + \mathbf{e}$.
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  To expand syndrome, assume $H = (Q^\mathsf{T} | I_{n-k})$.
  Then $\mathbf{x} = (00\ldots0)||\mathbf{s}$ satisfies $\mathbf{s} = H\mathbf{x}$.
- Note that this $\mathbf{x}$ is not a solution to the syndrome decoding problem, unless it has very low weight.

The Niederreiter cryptosystem I

Developed in 1986 by Harald Niederreiter as a variant of the McEliece cryptosystem. This is the schoolbook version.

- Use $n \times n$ permutation matrix P and $n-k \times n-k$ invertible matrix S.
- Public Key: a scrambled parity-check matrix $K = SHP \in \mathbb{F}_2^{(n-k)\times n}$.
- Encryption: The plaintext $\mathbf{e}$ is an n-bit vector of weight t. The ciphertext $\mathbf{s}$ is the $(n-k)$-bit vector

$$\mathbf{s} = K\mathbf{e}.$$

- Decryption: Find a n-bit vector $\mathbf{e}$ with $\mathrm{wt}(\mathbf{e}) = t$ such that $\mathbf{s} = K\mathbf{e}$.
- The passive attacker is facing a t-error correcting problem for the public key, which seems to be random.

The Niederreiter cryptosystem II

- Public Key: a scrambled parity-check matrix $K = SHP$.
- Encryption: The plaintext $\mathbf{e}$ is an n-bit vector of weight t. The ciphertext $\mathbf{s}$ is the $(n-k)$-bit vector

$$\mathbf{s} = K\mathbf{e}.$$

- Decryption using secret key: Compute

$$\begin{aligned} S^{-1}\mathbf{s} &= S^{-1}K\mathbf{e} = S^{-1}(SHP)\mathbf{e} \\ &= H(P\mathbf{e}) \end{aligned}$$

and observe that $\mathrm{wt}(P\mathbf{e}) = t$, because P permutes. Use efficient syndrome decoder for H to find $\mathbf{e}' = P\mathbf{e}$ and thus $\mathbf{e} = P^{-1}\mathbf{e}'$.

Note on codes

- McEliece proposed to use binary Goppa codes.
  These are still used today.
- Niederreiter described his scheme using Reed-Solomon codes.
  These were broken in 1992 by Sidelnikov and Chestakov.
- More corpses on the way: concatenated codes, Reed-Muller codes, several Algebraic Geometry (AG) codes, Gabidulin codes, several LDPC codes, cyclic codes.
- Some other constructions look OK (for now).
  NIST competition has several entries on QCMDPC codes.

Binary Goppa code

Let $q = 2^m$. A binary Goppa code is often defined by

- a list $L = (a_1, \ldots, a_n)$ of n distinct elements in $\mathbb{F}_q$, called the support.
- a square-free polynomial $g(x) \in \mathbb{F}_q[x]$ of degree t such that $g(a) \neq 0$ for all $a \in L$. $g(x)$ is called the Goppa polynomial.
- E.g. choose $g(x)$ irreducible over $\mathbb{F}_q$.

The corresponding binary Goppa code $\Gamma(L, g)$ is

$$\left\{ \mathbf{c} \in \mathbb{F}_2^n \,\middle|\, S(\mathbf{c}) = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2} + \cdots + \frac{c_n}{x - a_n} \equiv 0 \bmod g(x) \right\}$$

- This code is linear $S(\mathbf{b} + \mathbf{c}) = S(\mathbf{b}) + S(\mathbf{c})$ and has length n.
- What can we say about the dimension and minimum distance?

Dimension of $\Gamma(L, g)$

- $g(a_i) \neq 0$ implies $\gcd(x - a_i, g(x)) = 1$, thus get polynomials

$$(x - a_i)^{-1} \equiv f_i(x) \equiv \sum_{j=0}^{t-1} f_{i,j} x^j \bmod g(x)$$

  via XGCD. All this is over $\mathbb{F}_q = \mathbb{F}_{2^m}$.
- In this form, $S(\mathbf{c}) \equiv 0 \bmod g(x)$ means

$$\sum_{i=1}^{n} c_i \left(\sum_{j=0}^{t-1} f_{i,j} x^j \right) = \sum_{j=0}^{t-1} \left(\sum_{i=1}^{n} c_i f_{i,j} \right) x^j = 0,$$

  meaning that for each $0 \leq j \leq t - 1$:

$$\sum_{i=1}^{n} c_i f_{i,j} = 0.$$

- These are t conditions over $\mathbb{F}_q$, so tm conditions over $\mathbb{F}_2$. Giving an $tm \times n$ parity check matrix over $\mathbb{F}_2$.
- Some rows might be linearly dependent, so $k \geq n - tm$.

Nice parity check matrix

Assume $g(x) = \sum_{i=0}^{t} g_i x^i$ monic, i.e., $g_t = 1$.

$$H = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ g_{t-1} & 1 & 0 & \dots & 0 \\ g_{t-2} & g_{t-1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & g_3 & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{t-1} & a_2^{t-1} & a_3^{t-1} & \dots & a_n^{t-1} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{1}{g(a_1)} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{g(a_2)} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{g(a_3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{g(a_n)} \end{pmatrix}$$

Minimum distance of $\Gamma(L, g)$. Put $s(x) = S(\mathbf{c})$

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$$\begin{aligned} s(x) &= \sum_{i=1}^{n} c_i/(x - a_i) \\ &= \left(\sum_{i=1}^{n} c_i \prod_{j \neq i} (x - a_j) \right) / \prod_{i=1}^{n} (x - a_i) \equiv 0 \bmod g(x). \end{aligned}$$

- $g(a_i) \neq 0$ implies $\gcd(x - a_i, g(x)) = 1$,
  so $g(x)$ divides $\sum_{i=1}^{n} c_i \prod_{j \neq i} (x - a_j)$.
- Let $\mathbf{c} \neq 0$ have small weight $\mathrm{wt}(\mathbf{c}) = w \leq t = \deg(g)$.
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- Let $\mathbf{c} \neq 0$ have small weight $\mathrm{wt}(\mathbf{c}) = w \leq t = \deg(g)$. For all i with $c_i = 0$, $x - a_i$ appears in every summand. Cancel out those $x - a_i$ with $c_i = 0$.
- The denominator is now $\prod_{i, c_i \neq 0} (x - a_i)$, of degree w.
- The numerator now has degree $w - 1$ and $\deg(g) > w - 1$ implies that the numerator is $= 0$ (without reduction mod g), which is a contradiction to $\mathbf{c} \neq 0$, so $\mathrm{wt}(\mathbf{c}) = w \geq t + 1$.

Better minimum distance for $\Gamma(L, g)$

- Let $\mathbf{c} \neq 0$ have small weight $\mathrm{wt}(\mathbf{c}) = w$.
- Put $f(x) = \prod_{i=1}^{n}(x - a_i)^{c_i}$ with $c_i \in \{0, 1\}$.
- Then the derivative $f'(x) = \sum_{i=1}^{n} c_i \prod_{j \neq i}(x - a_i)^{c_i}$.
- Thus $s(x) = f'(x)/f(x) \equiv 0 \bmod g(x)$.
- As before this implies $g(x)$ divides the numerator $f'(x)$.
- Note that over $\mathbb{F}_{2^m}$:

$$(f_{2i+1}x^{2i+1})' = f_{2i+1}x^{2i}, \quad (f_{2i}x^{2i})' = 0 \cdot f_{2i}x^{2i-1} = 0,$$

  thus $f'(x)$ contains only terms of even degree and $\deg(f') \leq w - 1$. Assume w odd, thus $\deg(f') = w - 1$.
- Note that over $\mathbb{F}_{2^m}$: $(x + 1)^2 = x^2 + 1$

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$$(f_{2i+1}x^{2i+1})' = f_{2i+1}x^{2i}, \quad (f_{2i}x^{2i})' = 0 \cdot f_{2i}x^{2i-1} = 0,$$

  thus $f'(x)$ contains only terms of even degree and $\deg(f') \leq w - 1$. Assume w odd, thus $\deg(f') = w - 1$.
- Note that over $\mathbb{F}_{2^m}$: $(x + 1)^2 = x^2 + 1$ and in general

$$f'(x) = \sum_{i=0}^{(w-1)/2} f_{2i+1}x^{2i} = \left(\sum_{i=0}^{(w-1)/2} \sqrt{f_{2i+1}}x^i \right)^2 = F^2(x).$$

- Since $g(x)$ is square-free, $g(x)$ divides $F(x)$, thus $w \geq 2t + 1$.

Decoding of $\mathbf{c} + \mathbf{e}$ in $\Gamma(L, g)$

- Decoding works with polynomial arithmetic.
- Fix $\mathbf{e}$. Let $\sigma(x) = \prod_{i, e_i \neq 0}(x - a_i)$. Same as $f(x)$ before for $\mathbf{c}$.
- $\sigma(x)$ is called error locator polynomial. Given $\sigma(x)$ can factor it to retrieve error positions, $\sigma(a_i) = 0 \Leftrightarrow$ error in i.
- Split into odd and even terms: $\sigma(x) = A^2(x) + xB^2(x)$.
- Note as before $s(x) = \sigma'(x)/\sigma(x)$ and $\sigma'(x) = B^2(x)$.
- Thus

$$B^2(x) \equiv \sigma(x)s(x) \equiv (A^2(x) + xB^2(x))s(x) \bmod g(x)$$

$$B^2(x)(x + 1/s(x)) \equiv A^2(x) \bmod g(x)$$

- Put $v(x) \equiv \sqrt{x + 1/s(x)} \bmod g(x)$, then $A(x) \equiv B(x)v(x) \bmod g(x)$.
- Can compute $v(x)$ from $s(x)$.
- Use XGCD on v and g, stop part-way when

$$A(x) = B(x)v(x) + h(x)g(x),$$

with $\deg(A) \leq \lfloor t/2 \rfloor, \deg(B) \leq \lfloor (t-1)/2 \rfloor$.

Reminder: How to hide nice code?

- Do not reveal matrix H related to nice-to-decode code.
- Pick a random invertible $(n-k) \times (n-k)$ matrix S and random $n \times n$ *permutation matrix* P. Put

$$K = SHP.$$

- K is the public key and S and P together with a decoding algorithm for H form the private key.
- For suitable codes K looks like random matrix.
- How to decode syndrome $\mathbf{s} = K\mathbf{e}$?

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- How to decode syndrome $\mathbf{s} = K\mathbf{e}$?
- Computes $S^{-1}\mathbf{s} = S^{-1}(SHP)\mathbf{e} = H(P\mathbf{e})$.
- P permutes, thus $P\mathbf{e}$ has same weight as $\mathbf{e}$.
- Decode to recover $P\mathbf{e}$, then multiply by P^{-1}.

How to hide nice code?

- For Goppa code use secret polynomial $g(x)$.
- Use secret permutation of the a_i, this corresponds to secret permutation of the n positions; this replaces P.
- Use systematic form $K = (K'|I)$ for key;
  - This implicitly applies S.
  - No need to remember S because decoding does not use H.
  - Public key size decreased to $(n-k) \times k$.
- Secret key is polynomial g and support $L = (a_1, \ldots, a_n)$.

McBits (Bernstein, Chou, Schwabe, CHES 2013)

- Encryption is super fast anyways (just a vector-matrix multiplication).
- Main step in decryption is decoding of Goppa code. The McBits software achieves this in constant time.
- Decoding speed at 2^{128} pre-quantum security: $(n; t) = (4096; 41)$ uses 60493 Ivy Bridge cycles.
- Decoding speed at 2^{263} pre-quantum security: $(n; t) = (6960; 119)$ uses 306102 Ivy Bridge cycles.
- Grover speedup is less than halving the security level, so the latter parameters offer at least 2^{128} post-quantum security.
- More at `https://binary.cr.yp.to/mcbits.html`.

Do not use the schoolbook versions!

Sloppy Alice attacks! 1998 Verheul, Doumen, van Tilborg

- Assume that the decoding algorithm decodes up to t errors, i. e. it decodes $\mathbf{y} = \mathbf{c} + \mathbf{e}$ to $\mathbf{c}$ if $\mathrm{wt}(\mathbf{e}) \leq t$.
- Eve intercepts ciphertext $\mathbf{y} = \mathbf{m}G' + \mathbf{e}$.
  Eve poses as Alice towards Bob and sends him tweaks of $\mathbf{y}$.
  She uses Bob's reactions (success of failure to decrypt) to recover $\mathbf{m}$.
- Assume $\mathrm{wt}(\mathbf{e}) = t$. (Else flip more bits till Bob fails).
- Eve sends $\mathbf{y}_i = \mathbf{y} + \mathbf{e}_i$ for $\mathbf{e}_i$ the i-th unit vector.
  If Bob returns error, position i in $\mathbf{e}$ is 0 (so the number of errors has increased to $t + 1$ and Bob fails).
  Else position i in $\mathbf{e}$ is 1.
- After k steps Eve knows the first k positions of $\mathbf{m}G'$ without error. Invert the $k \times k$ submatrix of G' to get $\mathbf{m}$

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- After k steps Eve knows the first k positions of $\mathbf{m}G'$ without error. Invert the $k \times k$ submatrix of G' to get $\mathbf{m}$ assuming it is invertible.
- Proper attack: figure out invertible submatrix of G' at beginning; recover matching k coordinates.

More on sloppy Alice

- This attack has Eve send Bob variations of the same ciphertext; so Bob will think that Alice is sloppy.
- Note, this is more complicated if $\mathbb{F}_q$ instead of $\mathbb{F}_2$ is used.
- Other name: reaction attack.
  (1999 Hall, Goldberg, and Schneier)
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- Attack also works on Niederreiter version:
  Bitflip cooresponds to sending $\mathbf{s}_i = \mathbf{s} + K_i$,
  where K_i is the i-th column of K.
- More involved but doable (for McEliece and Niederreiter) if decryption requires exactly t errors.

Berson's attack

- Eve knows $\mathbf{y}_1 = \mathbf{m}G' + \mathbf{e}_1$ and $\mathbf{y}_2 = \mathbf{m}G' + \mathbf{e}_2$; these have the same $\mathbf{m}$.

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- If $\mathrm{wt}(\bar{\mathbf{e}}) = 2t$:
  All zero positions in $\bar{\mathbf{e}}$ are error free in both ciphertexts.
  Invert G' in those columns to recover $\mathbf{m}$ as in previous attack.
- Else: ignore the $2w = \mathrm{wt}(\bar{\mathbf{e}}) < 2t$ positions in G' and $\mathbf{y}_1$.
  Solve decoding problem for $k \times (n - 2w)$ generator matrix G''
  and vector $\mathbf{y}'_1$ with $t - w$ errors; typically much easier.

Formal security notions

- McEliece/Niederreiter are One-Way Encryption (OWE) schemes.
- However, the schemes as presented are not CCA–II secure:
  - Given challenge $\mathbf{y} = \mathbf{m}G' + \mathbf{e}$, Eve can ask for decryptions of anything but $\mathbf{y}$.

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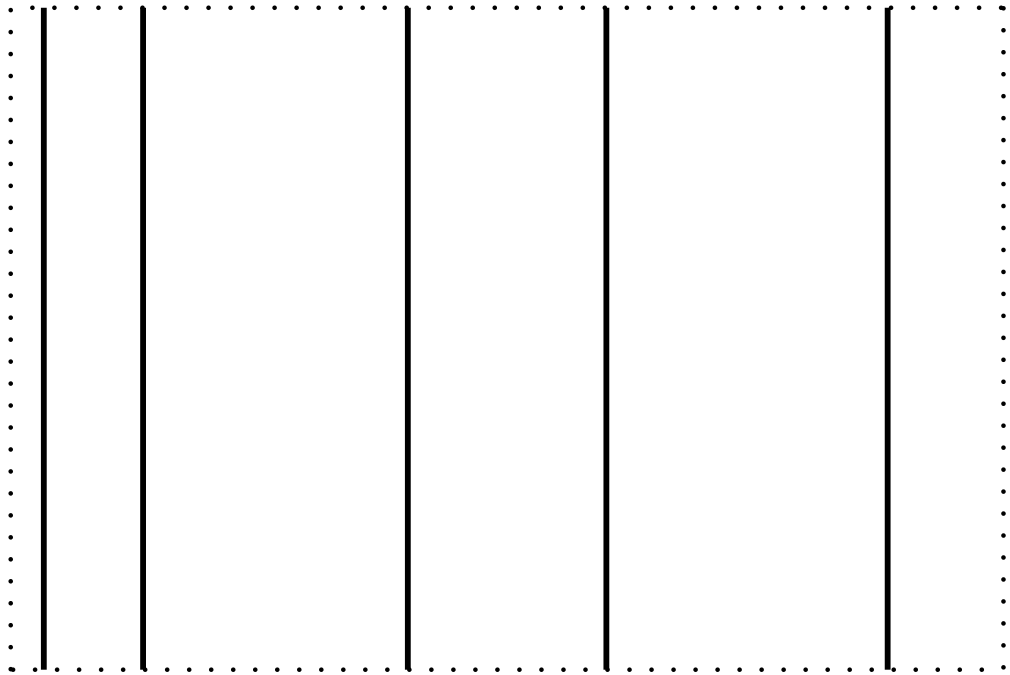
  - This is different from challenge $\mathbf{y}$, so Bob answers.
  - Answer is $\mathbf{m} + \bar{\mathbf{m}}$.
- Fix by using CCA2 transformation (e.g. Fujisaki-Okamoto transform) or (easier) KEM/DEM version: pick random $\mathbf{e}$ of weight t, use hash($\mathbf{e}$) as secret key to encrypt and authenticate (for McEliece or Niederreiter).

Generic attack: Brute force

Given K and $\mathbf{s} = K\mathbf{e}$, find $\mathbf{e}$ with $\mathrm{wt}(\mathbf{e}) = t$.

$K =$ 

Pick any group of t columns of K, add them and compare with $\mathbf{s}$.

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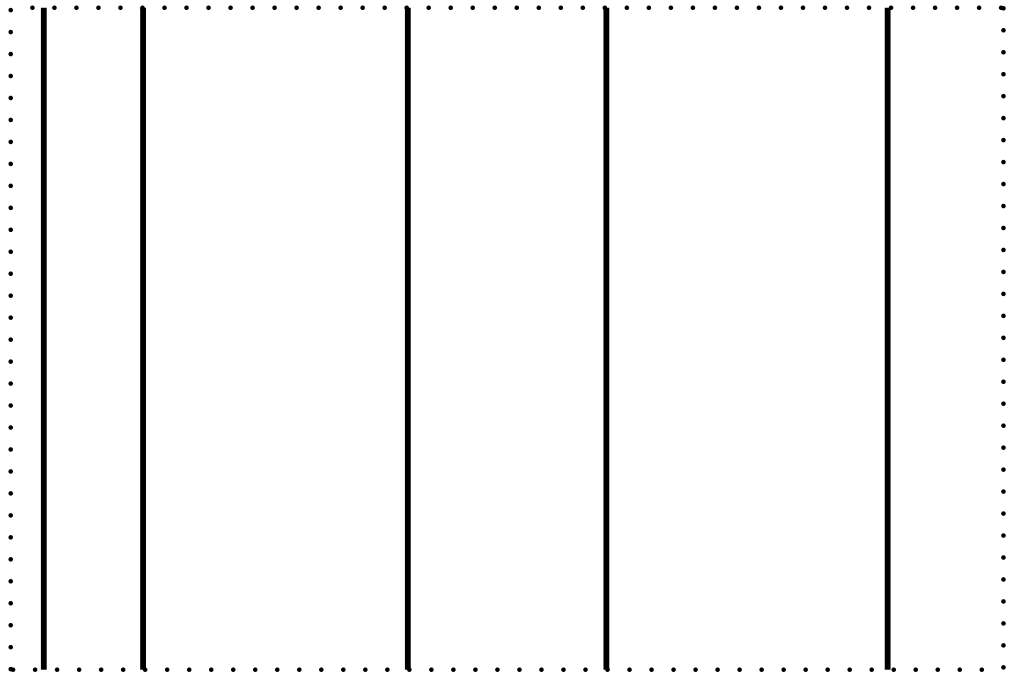
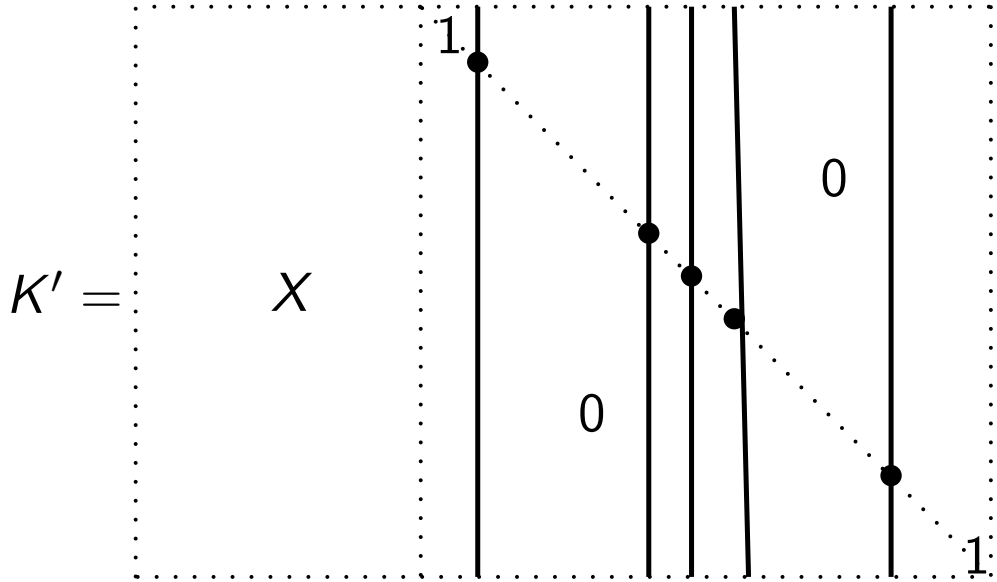
$K =$ 

Pick any group of t columns of K, add them and compare with $\mathbf{s}$.

Cost: $\binom{n}{t}$ sums of t columns.
Can do better so that each try costs only 1 column addition (after some initial additions).
Cost: $O\binom{n}{t}$ additions of 1 column.

Generic attack: Information-set decoding, 1962 Prange



1. Permute K and bring to systematic form $K' = (X|I_{n-k})$.
   (If this fails, repeat with other permutation).
2. Then $K' = UKP$ for some permutation matrix P and U the matrix that produces systematic form.
3. This updates $\mathbf{s}$ to $U\mathbf{s}$.
4. If $\mathrm{wt}(U\mathbf{s}) = t$ then $\mathbf{e}' = (00\ldots0)||U\mathbf{s}$.
   Output unpermuted version of $\mathbf{e}'$.
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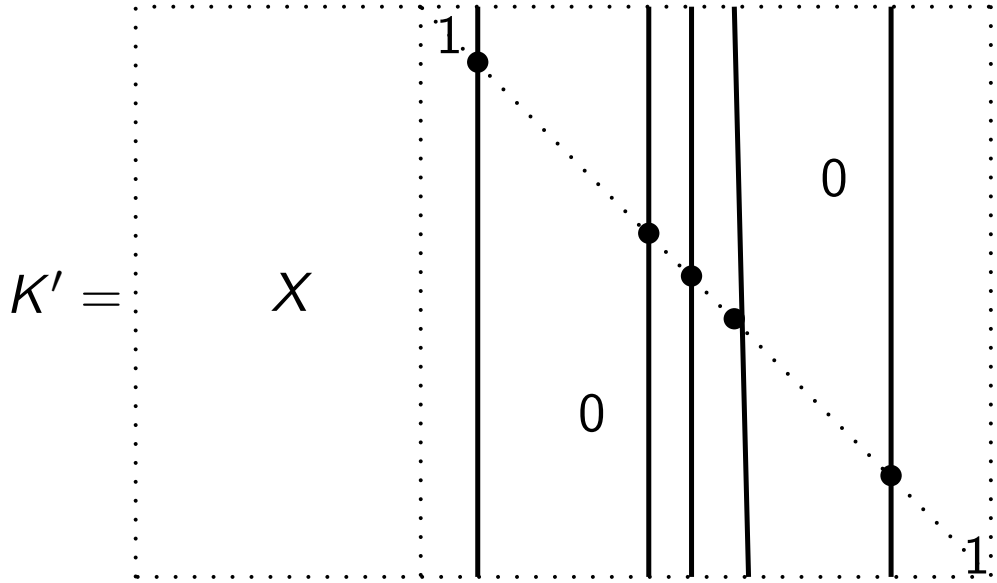

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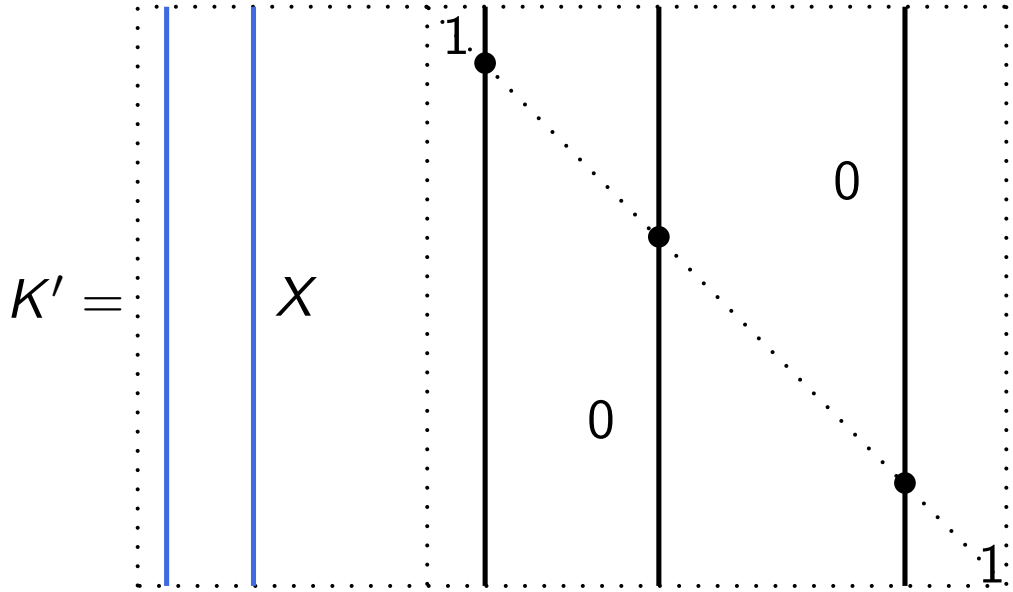
Cost: $O(\binom{n}{t}/\binom{n-k}{t})$ matrix operations.

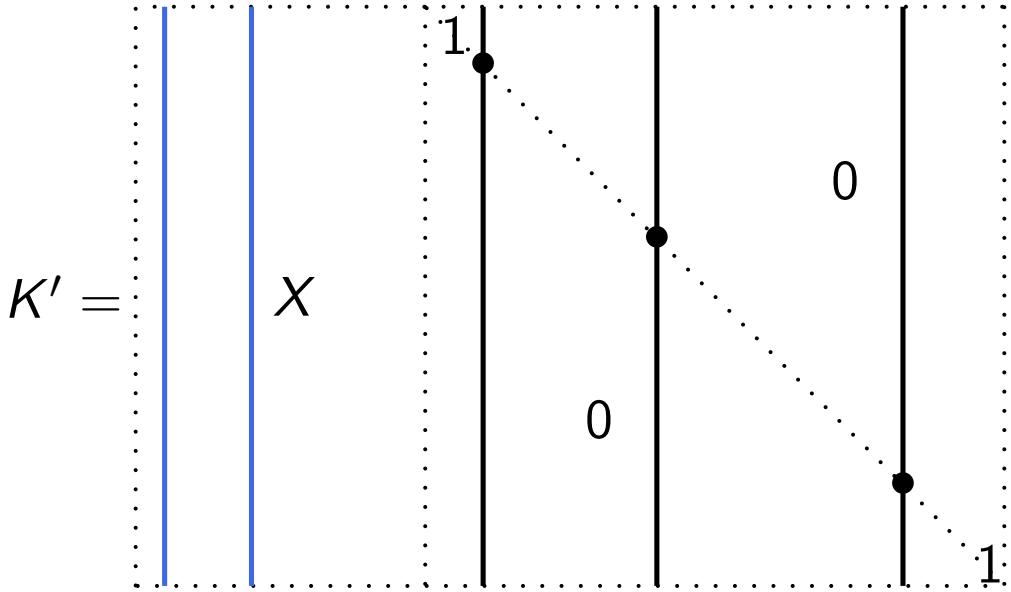
Lee–Brickell attack



1. Permute K and bring to systematic form $K' = (X|I_{n-k})$. (If this fails, repeat with other permutation). $\mathbf{s}$ is updated.
2. For small p, pick p of the k columns on the left, compute their sum $X\mathbf{p}$. ($\mathbf{p}$ is the vector of weight p).
3. If $\mathrm{wt}(\mathbf{s} + X\mathbf{p}) = t - p$ then put $\mathbf{e}' = \mathbf{p}||(\mathbf{s} + X\mathbf{p})$. Output unpermuted version of $\mathbf{e}'$.
4. Else return to 2 or return to 1 to rerandomize.

Cost:

Lee–Brickell attack



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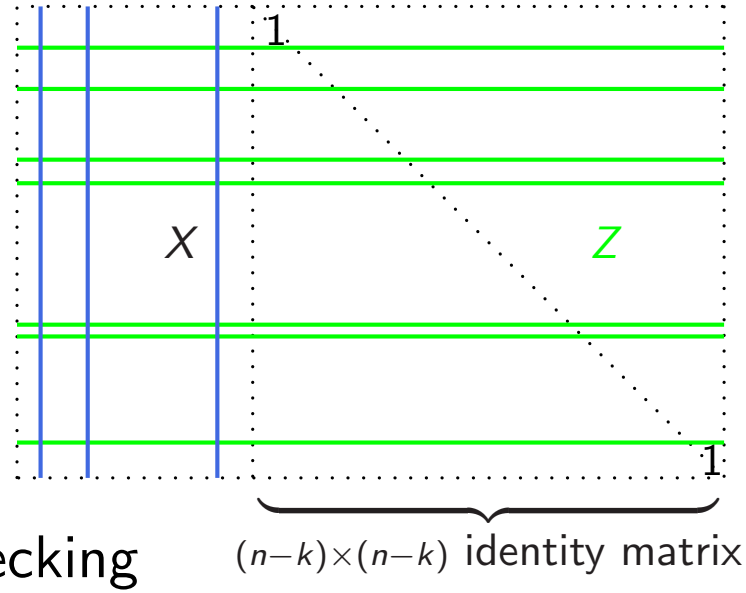
4. Else return to 2 or return to 1 to rerandomize.

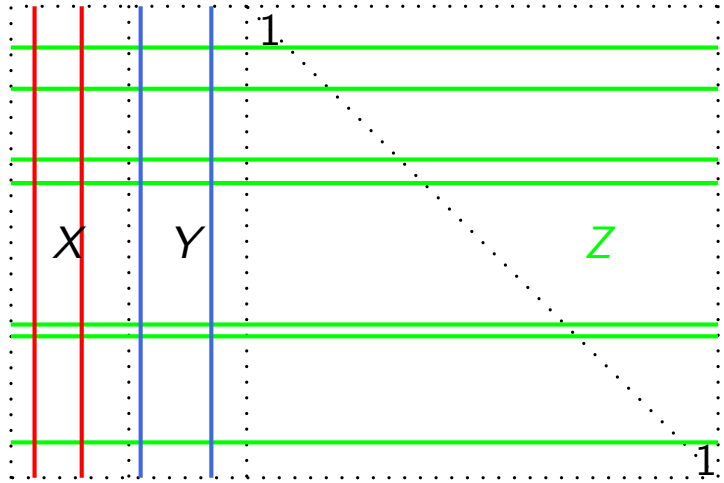
Cost: $O(\binom{n}{t}/(\binom{k}{p}\binom{n-k}{t-p}))$ [matrix operations$+\binom{k}{p}$ column additions].

Leon's attack



- Setup similar to Lee-Brickell's attack.
- Random combinations of p vectors will be dense, so have $\mathrm{wt}(\mathbf{s} + X\mathbf{p}) \sim k/2$.
- Idea: Introduce early abort by checking only ℓ positions (selected by set Z, green lines in the picture). This forms $\ell \times k$ matrix X_Z, length-ℓ vector $\mathbf{s}_Z$.
- Inner loop becomes:
  1. Pick $\mathbf{p}$ with $\mathrm{wt}(\mathbf{p}) = p$.
  2. Compute $X_Z\mathbf{p}$.
  3. If $\mathbf{s}_Z + X_Z\mathbf{p} \neq 0$ goto 1.
  4. Else compute $X\mathbf{p}$.
     - 4.1 If $\mathrm{wt}(\mathbf{s} + X\mathbf{p}) = t - p$ then put $\mathbf{e}' = \mathbf{p}||(\mathbf{s} + X\mathbf{p})$. Output unpermuted version of $\mathbf{e}'$.
     - 4.2 Else return to 1 or rerandomize K.
- Note that $\mathbf{s}_Z + X_Z\mathbf{p} = 0$ means that there are no ones in the positions specified by Z. Small loss in success, big speedup.

Stern's attack



- Setup similar to Leon's and Lee-Brickell's attacks.
- Use the early abort trick, so specify set Z.
- Improve chances of finding $\mathbf{p}$ with $\mathbf{s} + X_Z\mathbf{p} = 0$:
  - Split left part of K' into two disjoint subsets X and Y.
  - Let $A = \{\mathbf{a} \in \mathbb{F}_2^{k/2} | \mathrm{wt}(\mathbf{a}) = p\}$, $B = \{\mathbf{b} \in \mathbb{F}_2^{k/2} | \mathrm{wt}(\mathbf{b}) = p\}$.
  - Search for words having exactly p ones in X and p ones in Y and exactly $w - 2p$ ones in the remaining columns.
  - Do the latter part as a collision search:
    Compute $\mathbf{s}_Z + X_Z\mathbf{a}$ for all (many) $\mathbf{a} \in A$, sort.
    Then compute $Y_Z\mathbf{b}$ for $\mathbf{b} \in B$ and look for collisions; expand.
  - Iterate until word with $\mathrm{wt}(\mathbf{s} + X\mathbf{a} + Y\mathbf{b}) = 2p$ is found for some X, Y, Z.
- Select p, ℓ, and the subset of A to minimize overall work.

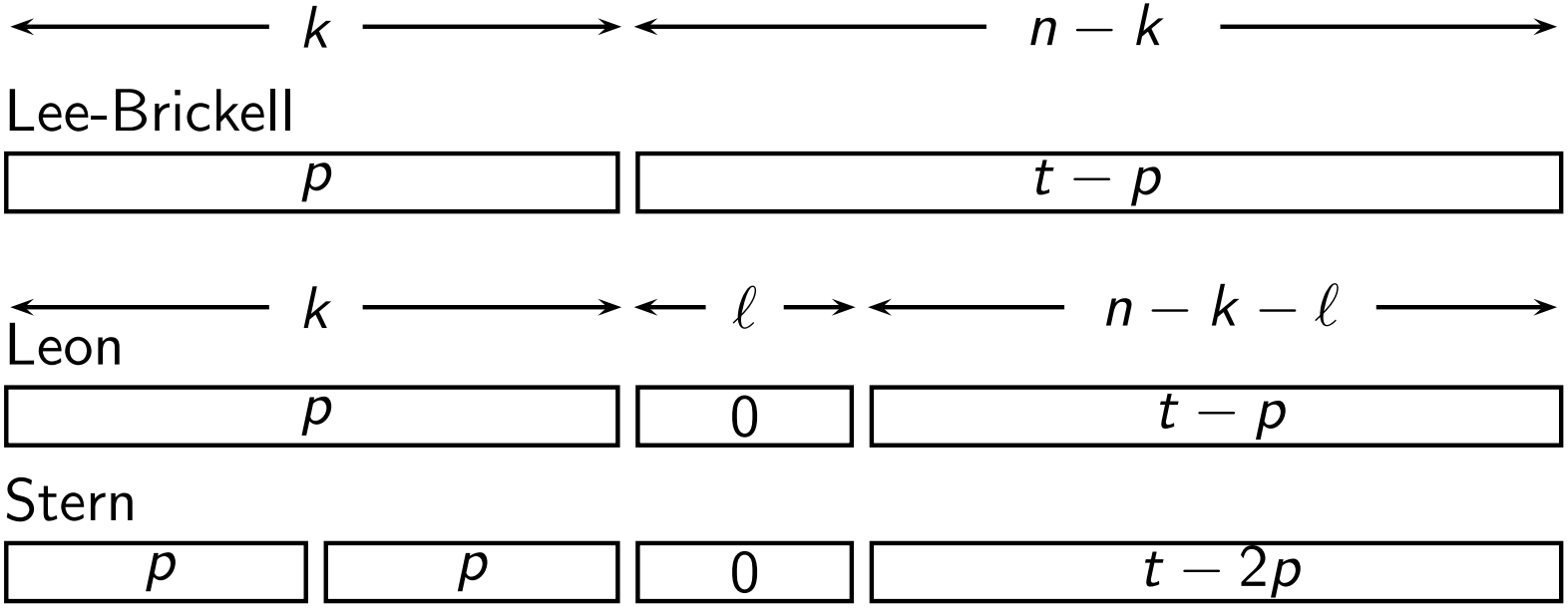
Running time in practice

2008 Bernstein, Lange, Peters.

- Wrote attack software against original McEliece parameters, decoding 50 errors in a $[1024, 524]$ code.
- Lots of optimizations, e.g. cheap updates between $\mathbf{s}_Z + X_Z\mathbf{a}$ and next value for $\mathbf{a}$; optimized frequency of K randomization.
- Attack on a single computer with a 2.4GHz Intel Core 2 Quad Q6600 CPU would need, on average, 1400 days (2^{58} CPU cycles) to complete the attack.
- About 200 computers involved, with about 300 cores.
- Most of the cores put in far fewer than 90 days of work; some of which were considerably slower than a Core 2.
- Computation used about 8000 core-days.
- Error vector found by Walton cluster at SFI/HEA Irish Centre of High-End Computing (ICHEC).

Information-set decoding

Methods differ in where the "errors" are allowed to be.



Running time is exponential for Goppa parameters n, k, d.

Information-set decoding

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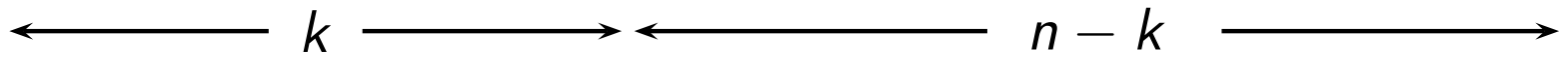
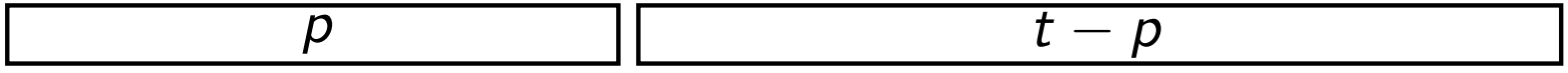
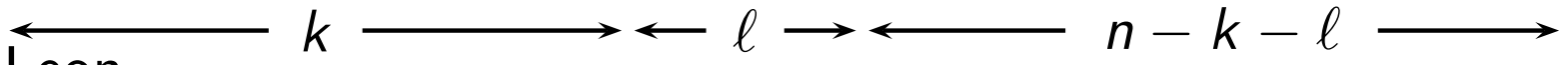
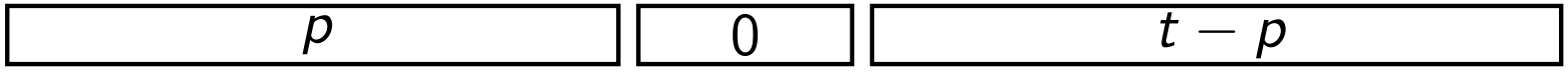


Lee-Brickell

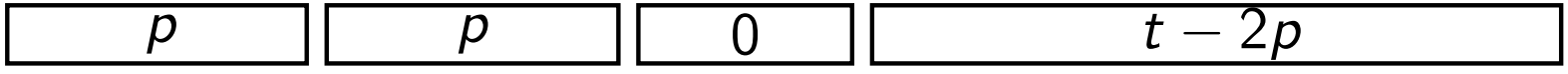




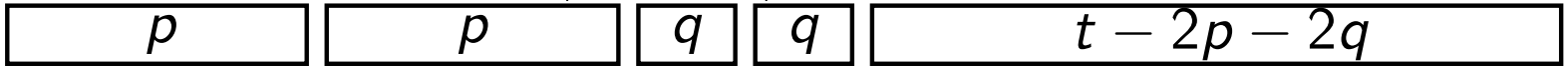
Leon

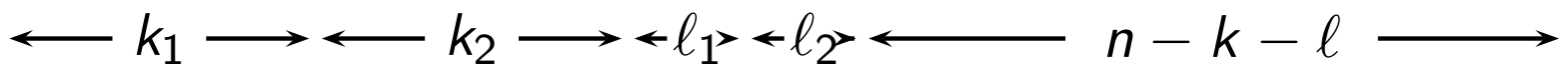


Stern



Ball-collision decoding/Dumer/Finiasz-Sendrier





2011 May-Meurer-Thomae and 2012 Becker-Joux-May-Meurer refine multi-level collision search. No change in exponent for Goppa parameters n, k, d.

Improvements

- Increase n: The most obvious way to defend McEliece's cryptosystem is to increase the code length n.
- Allow values of n between powers of 2: Get considerably better optimization of (e.g.) the McEliece public-key size.
- Use list decoding to increase t: Unique decoding is ensured by CCA2-secure variants.
- Decrease key size by using fields other than $\mathbb{F}_2$ (wild McEliece).
- Decrease key size & be faster by using other codes. Needs security analysis: some codes have too much structure.

More exciting codes

- We distinguish between generic attacks (such as information-set decoding) and structural attacks (that use the structure of the code).
- Gröbner basis computation is a generally powerful tool for structural attacks.
- Cyclic codes need to store only top row of matrix, rest follows by shifts. Quasi-cyclic: multiple cyclic blocks.
- QC Goppa: too exciting, too much structure.
- Interesting candidate: Quasi-cyclic Moderate-Density Parity-Check (QC-MDPC) codes, due to Misoczki, Tillich, Sendrier, and Barreto (2012).
  Very efficient but practical problem if the key is reused (Asiacrypt 2016).
- Hermitian codes, general algebraic geometry codes.
- Please help us update `https://pqcrypto.org/code.html`.

Bonus slides

RaCoSS – Random Code-based Signature Schemes

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- System parameters: $n = 2400$, $k = 2060$.
  Random matrix $H \in \mathbb{F}_2^{(n-k)\times n}$.
- Secret key: sparse $S \in \mathbb{F}_2^{n\times n}$.
- Public key: $T = H \cdot S$. (looks pretty random).
- Sign m: Pick a low weight $y \in \mathbb{F}_2^n$.
  Compute $v = Hy$, $c = h(v, m)$, $z = Sc + y$. Output (z, c).

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  Compute $v = Hy$, $c = h(v, m)$, $z = Sc + y$. Output (z, c).
- Verify $m, (z, c)$: Check that $\mathrm{weight}(z) \leq 1564$.
  Compute $v' = Hz + Tc$. Check that $h(v', m) = c$.

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This needs a special hash function so that c is very sparse.

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Implementation bug:

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...compares only the first 300 coefficients!
Thus, a signature with $c[0...299] = 0$ is accepted for

$$\binom{2100}{3} / \binom{2400}{3} \approx 67\%$$

of all messages.

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possible outputs.
- Slow: 600 to 800 hashes per second and core.
- Expected time for a preimage on ≈ 100 cores: 10 hours.
- crashed while brute-forcing: *memory leaks*

- another message signed by the first KAT:

> NISTPQC is so much fun! 10900qmmP

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  Pick $n - k$ columns of H that form an invertible matrix H_1.
- Compute $z = (z_1 || 00 \ldots 0)$ by linear algebra.
- Expected weight of z is $\approx (n - k)/2 = 170 \ll 1564$.
- Properly generated signatures have weight$(z) \approx 261$.

RaCoSS – Summary

- Bug in code: bit vs. byte confusion meant only every 8th bit verified.
- Preimages for RaCoSS' special hash function: only

$$\binom{2400}{3} = 2301120800 \sim 2^{31.09}$$

  possible outputs.
- The code dimensions give a lot of freedom to the attacker – our forged signature is better than a real one!

Classic McEliece
conservative code-based cryptography

Daniel J. Bernstein, Tung Chou, Tanja Lange,
Ingo von Maurich, Rafael Misoczki, Ruben Niederhagen,
Edoardo Persichetti, Christiane Peters, Peter Schwabe,
Nicolas Sendrier, Jakub Szefer, Wen Wang

Key sizes and key-generation speed

`mceliece6960119` parameter set:

1047319 bytes for public key.

13908 bytes for secret key.

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1357824 bytes for public key.

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Very fast in hardware (PQCrypto 2018; CHES 2017):
a few million cycles at 231MHz
using 129059 modules, 1126 RAM blocks
on Altera Stratix V FPGA.

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Can tweak parameters for even smaller ciphertexts, not much penalty in key size.

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Fundamental security question:
Given random parity-check matrix H and syndrome s,
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guiding sizes in 1978 McEliece.

The McEliece system (with later key-size optimizations)
uses $(c_0 + o(1))\lambda^2(\lg \lambda)^2$-bit keys as $\lambda \to \infty$
to achieve 2^λ security against Prange's attack.

Here $c_0 \approx 0.7418860694$.

40 years and more than 30 analysis papers later

1962 Prange; 1981 Clark–Cain, crediting Omura; 1988 Lee–Brickell; 1988 Leon; 1989 Krouk; 1989 Stern; 1989 Dumer; 1990 Coffey–Goodman; 1990 van Tilburg; 1991 Dumer; 1991 Coffey–Goodman–Farrell; 1993 Chabanne–Courteau; 1993 Chabaud; 1994 van Tilburg; 1994 Canteaut–Chabanne; 1998 Canteaut–Chabaud; 1998 Canteaut–Sendrier; 2008 Bernstein–Lange–Peters; 2009 Bernstein–Lange–Peters–van Tilborg; 2009 Bernstein (**post-quantum**); 2009 Finiasz–Sendrier; 2010 Bernstein–Lange–Peters; 2011 May–Meurer–Thomae; 2012 Becker–Joux–May–Meurer; 2013 Hamdaoui–Sendrier; 2015 May–Ozerov; 2016 Canto Torres–Sendrier; 2017 Kachigar–Tillich (**post-quantum**); 2017 Both–May; 2018 Both–May; 2018 Kirshanova (**post-quantum**).

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Replacing λ with 2λ stops all known *quantum* attacks.

Classic McEliece

McEliece's system prompted huge amount of followup work.

Some work improves efficiency while clearly preserving security:

- Niederreiter's dual PKE
  (use parity check matrix instead of generator matrix);
- many decoding speedups; ...

Classic McEliece uses all this, with constant-time implementations.

- Write $H = (I_{n-k}|T)$, public key is $(n-k) \times k$ matrix T,
  $n - k = w \log_2 q$. H constructed from binary Goppa code.
- Encapsulate using e of weight w.

`mceliece6960119` parameter set (2008 Bernstein–Lange–Peters):
$q = 8192$, $n = 6960$, $w = 119$.

`mceliece8192128` parameter set:
$q = 8192$, $n = 8192$, $w = 128$.

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5. Ciphertext is deterministic function of input e: i.e., inversion recovers all randomness used to create ciphertexts.

6. There are no inversion failures for legitimate ciphertexts.

Classic McEliece highlights

- Security asymptotics unchanged by 40 years of cryptanalysis.
- Short ciphertexts.
- Efficient and straightforward conversion of OW-CPA PKE into IND-CCA2 KEM.
- Constant-time software implementations.
- FPGA implementation of full cryptosystem.
- Open-source (public domain) implementations.
- No patents.

Metric	**mceliece6960119**	**mceliece8192128**
Public-key size	1047319 bytes	1357824 bytes
Secret-key size	13908 bytes	14080 bytes
Ciphertext size	226 bytes	240 bytes
Key-generation time	1108833108 cycles	1173074192 cycles
Encapsulation time	153940 cycles	188520 cycles
Decapsulation time	318088 cycles	343756 cycles

See https://classic.mceliece.org for more details.